Jurnal Matematika Integratif. p-ISSN:1412-6184, e-ISSN:2549-903
Vol. 13, No. 1 (2017), pp. 48–51. doi:10.24198/jmi.v13.n1.11505.48-51

Biharmonic Equation on Annulus in a Unit Sphere with Polynomial Boundary Condition

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Abstract. We study the Biharmonic boundary value problem on the annulus with polynomial boundary condition. We utilize an exact algorithms for solving Laplace equation with polynomial Dirichlet conditions with algorithm. The algorithm requires differentiation of the boundary function, but no integration.

Key words and Phrases: Annulus, Biharmonic equation, Dirichlet condition, Laplace equation, polynomial data.

1. INTRODUCTION

We have known that the harmonic function is a function that satisfies the Laplace equation. The Laplace equation is very important equation and has many applications in the real phenomena such as conductivity equation and wave equation. Biharmonic function is a function which satisfies Bi-Laplace problem. We are interested in studying the Biharmonic equation in this paper.

We study the Biharmonic equation on Annulus with polynomial data in $\mathbb{R}^3$. The reason why we use the polynomial data is because it can be writen as the linear combination of the harmonic functions (Axler and Ramey, 1995). We approached the solution by using Lemma that has been formulated by Axler and Ramey (1995) and Lemma by Herzog (2000). In the previous paper, we have studied about Biharmonic in a unit ball in $\mathbb{R}^3$ with polynomial data, Maulidi and Garnadi (2016).

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, thus for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of non-negative integers we say x^α as monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$. The degree of x^α is $\alpha_1+\alpha_2+\alpha_3$. A polynomial is said to be homogeneous of degree m if it is a finite linear combination of monomial x^α of degree m ; $m = 0, 1, 2, \ldots$. Let P_m denotes the vector space of polynomials in $\mathbb{R}^3$, homogeneous of degree m, and H_m is the subspace of harmonic polynomials of degree m, where it is satisfies the Laplace equation, then we have property by Axler and Ramey as follows

$$P_m = H_m + |x|^2 P_{m-2},$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Let Ω is a unit sphere in $\mathbb{R}^3$, with the boundary

$$\Gamma = \partial\Omega = \{x \in \mathbb{R}^3 | \quad x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Let also

$$D = \{x \in \mathbb{R}^3 \quad ; 0 \le |x| \le R < 1\},$$

then $D^o = \Omega - D$ is an annulus region in $\mathbb{R}^3$.

The Biharmonic on Annulus problem can be formulated as follows:

$$\Delta^2 u = 0 \text{ in } D^o, \tag{1}$$

with boundary condition

$$u|_\Gamma = p_1; \qquad u|_D = p_2,$$

and

$$\Delta u|_\Gamma = q_1; \qquad \Delta u|_D = q_2,$$

where, p_1, p_2, q_1 and q_2 are polynomials.

2. TECHNICAL LEMMAS

These following Lemmas are used to obtain the solution of problem (1)

Lemma 2.1. *If $p \in P_m$, then the solution to the Dirichlet problem in a unit sphere, $u \in P_m$, with boundary data $u|_\Gamma = p$ is*

$$p_m + p_{m-2} + ... + p_{m-2k} \tag{2}$$

where $k = [\frac{m}{2}]$ and $p_m, p_{m-2}, ..., p_{m-2k}$ are the harmonic polynomials.

The proof of Lemma 2.1 can be seen in Axler and Ramey (1995) and we recommend you to study the algorithm to obtain p_j which is a harmonic function.

Lemma 2.2. *Given $p \in P_m$ and $q \in P_{m+2}$ then there exists $u \in P_{m+2}$ such that*

$$\Delta u(x) = q(x) \qquad x \in \Omega \qquad \textit{and} \qquad u|_\Gamma = p(x).$$

Proof. see Herzog (2000).

Lemma 2.3. *Let $0 < r < s < \infty$, the harmonic function on the annular region in a unit sphere $\{r \leq |x| \leq s\}$ that equals $p \in P_m$ on $\{|x| = r\}$ and equals $q \in P_n$ on $\{|x| = s\}$ is*

$$\sum_{j=0}^{m} \frac{|x|^{-1-2j} - s^{-1-2j}}{r^{-1-2j} - s^{-1-2j}} r^{m-j} p_j + \sum_{j=0}^{n} \frac{|x|^{-1-2j} - r^{-1-2j}}{s^{-1-2j} - r^{-1-2j}} s^{n-j} q_j.$$

Proof. The idea for this solution comes from Chapter 10 of [1].

3. THE MAIN THEOREM

Teorema 3.1. *Given $p_1 \in P_{m_1}, p_2 \in P_{m_2}, q_1 \in P_{n_1}$, and $q_2 \in P_{n_2}$ then there exists $u_1 \in P_m$ and $u_2 \in P_n$, $m = max(m_1, m_2)$ and $n = max(n_1, n_2)$, such that $u = u_1 + u_2$ is the solution of (1).*

Proof. Here we give the algorithm to solve the problem (1). The Biharmonic type problem was presented in (1) can be written in two subproblem:

$$\Delta u = v \quad \text{in} \quad D_o; \tag{3}$$

with conditions

$$u|_\Gamma = p_1,$$
$$u|_D = p_2.$$

And

$$\Delta v = 0 \quad \text{in} \quad D_o, \tag{4}$$

with conditions

$$v|_\Gamma = q_1,$$
$$v|_D = q_2.$$

The solution of problem (3) can be formulated if we had solved the problem (4). The solution of problem (4), $v \in P_n$, $n = max(n_1, n_2)$ can be obtained by using Lemma 2.3 which is a harmonic polynomials function. By setting $r = R$ and $s = 1$, then we have

$$v = \sum_{j=0}^{n_1} \frac{|x|^{-1-2j} - 1}{R^{-1-2j} - 1} R^{n_1 - j} q_{1j} + \sum_{j=0}^{n_2} \frac{|x|^{-1-2j} - R^{-1-2j}}{1 - R^{-1-2j}} q_{2j},$$

where q_{1j} and q_{2j} are the harmonic polyomials which have degree j. $q_{1j}, q_{2j} \in H_j$.

By using the principle of linear superposition, suppose that the solution of (3) is $u = u_1 + u_2$, where u_1 is the solution of this problem:

$$\Delta u_1 = 0 \quad ; u_1|_\Gamma = p_1 \text{ and } u_1|_D = p_2, \tag{5}$$

and u_2 is the solution of this problem:

$$\Delta u_2 = v \quad ; u_2|_\Gamma = 0 \text{ and } u_2|_D = 0. \tag{6}$$

The solution of (5) can be obtained by using Lemma 2.3 which is the harmonic polynomials function. Where $u_1 \in P_m$, $m = max(m_1, m_2)$. Next, by solving problem (6) when the solution of this problem is guaranteed by Lemma 2.2, so we have $u_2 \in P_n$. Therefore this completes the proof of Theorem 3.1.

4. CONCLUSION

The Biharmonic on annulus in a unit sphere with polynomial boundary condition that presented in (1) has a solution. The solution can be obtained by using the algorithm and the existence of this solution is guaranteed by using Lemma 2.2 and Lemma 2.3.

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