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The Properties of Rough Submodule Over Rough Ring

Rara Gusti Rahmawati, Fitriani, Ahmad Faisol

Department of Mathematics, Universitas Lampung
Bandar Lampung, Indonesia
Email: gustirahmawati27@gmail.com, fitriani.1984@fmipa.unila.ac.id, ahmadfaisol@fmipa.unila.ac.id

Abstract
A pair of non-empty set U and equivalence relation R on U, denoted as (U, R), is called approximation space. Furthermore, equivalence classes form the construction of lower approximation and upper approximation. Let $X \subseteq U$, the lower approximation of X denoted by $\underline{X}$ and the upper approximation of X denoted by $\overline{X}$. A pair $Apr(X) = (\underline{X}, \overline{X})$ is a rough set if $\underline{X} \neq \overline{X}$. $Apr(X)$ is rough module if $Apr(X)$ satisfies some conditions. In this research, we investigate some characteristics of the rough module and rough submodule over rough ring. Furthermore, we construct examples of the rough module and the rough submodule on approximation space (U, R).
Keywords: *Approximation space, rough group, rough ring, rough module, rough submodule.*

1. Introduction

Rough set theory is a new mathematical approach introduced by Pawlak in 1982 to solve imperfect or ambiguous problems in information systems. In this approach, vagueness is expressed by the boundaries of a set. Some of the applications of rough set theory can be seen in [1], [2], [3], [4], and [5]. The relationship between rough sets and algebraic structures such as groups [6], subgroups [7], rings [8], and modules [9], has been widely discussed in various previous studies. Miao et al. discussed the rough groups, rough subgroups, and their properties [10]. Furthermore, Davvaz and Mahdavipour discussed the rough modules and submodules, followed by research conducted by Zhang et al., which discussed the rough module and its properties, as in various other studies [11]. Moreover, Zhang et al. give the applications of rough sets [12].

In 2013, Davvaz and Malekzadeh used the reference point in investigating the roughness in the module [13]. Then, Neelima and Isaac investigate roughness in semiprime ideals [14]. Furthermore, they provide the rough primary fuzzy ideals and rough anti-homomorphism [15]. Moreover, Isaac et al. introduce the rough G-modules [15], Nugraha et al. investigate roughness in groups [17], Jesmalar gives the properties of homomorphism of rough groups [18], Setyaningsih et al. provide the sub-exact sequence of rough groups [19], Agusfrianto gives the properties of rough rings [20], and Hafifullah et al. provide the V-coexact sequence of rough groups [21]. Furthermore, Ayuni et al. investigate the rough U-exact sequence of rough groups [22] and Dwiyanti et al. give the properties of the rough projective modules [23].

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In this research, we investigate the characteristics of the rough module and rough submodule over the rough ring. Furthermore, we construct examples of the rough module and the rough submodule on an approximation space (U, R).

2. Methods

The research method depends on equivalence relations, equivalence classes, upper and lower approximations, rough set, rough group, rough ring, rough module, and rough submodule. The following stages of the research are:

(1) construct the rough module;
(2) analyze the properties of the rough module;
(3) construct the rough submodule from the previously constructed rough module;
(4) analyze the properties of the rough submodule.

3. Results and Discussion

Before constructing the rough module, we recall the definition of the rough module over the rough ring as follows.

Definition 3.1. *Let $(Apr(R), +, *)$ be a rough ring with a unit, $(Apr(M), +)$ a commutative rough group. $Apr(M)$ is called a rough module over the ring $Apr(R)$ if there is a mapping $\overline{R} \times \overline{M} \to \overline{M}, (a, x) \to ax$ such that*

(1) $a(x + y) = ax + ay, a \in Apr(R); x, y \in Apr(M)$;
(2) $(a + b)x = ax + bx, a, b \in Apr(R); x \in Apr(M)$;
(3) $(ab)x = a(bx), a, b \in Apr(R); xinApr(M)$;
(4) $1x = x$ [11].

Example 3.2. *Let $U = \mathbb{Z}_{20} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \dots, \overline{19}\}$. We define aRb if only if $a - b = 6z$ for $z \in \mathbb{Z}$. We have six equivalence classes in the following table based on the equivalence relation.*

Table 1. The equivalence classes of $\mathbb{Z}_{20}$

The Equivalence Classes	*The Element*
E_1	$\{\overline{0}, \overline{6}, \overline{12}, \overline{18}\}$
E_2	$\{\overline{1}, \overline{7}, \overline{13}, \overline{19}\}$
E_3	$\{\overline{2}, \overline{8}, \overline{14}\}$
E_4	$\{\overline{3}, \overline{9}, \overline{15}\}$
E_5	$\{\overline{4}, \overline{10}, \overline{16}\}$
E_6	$\{\overline{5}, \overline{11}, \overline{17}\}$

Furthermore, let $X = \{\overline{0}, \overline{1}, \overline{2}, \overline{4}, \overline{5}, \overline{8}, \overline{10}, \overline{12}, \overline{15}, \overline{16}, \overline{18}, \overline{19}\}$ a non-empty subset of U, $X \subseteq U$. In the approximation space (U, R), the lower and upper approximations of X are obtained as follows:
$\underline{X} = \{x | [x]_R \subseteq X\} = \emptyset$;
$\overline{X} = \{x | [x]_R \cap X \neq \emptyset\} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 = U$.

Next, we define the binary operation $+_{20}$ on U. We will show that $Apr(X)$ is rough group.

Table 2. Table Cayley $+_{20}$ on X

$+_{20}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{5}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{15}$	$\overline{16}$	$\overline{18}$	$\overline{19}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{5}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{15}$	$\overline{16}$	$\overline{18}$	$\overline{19}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{11}$	$\overline{13}$	$\overline{16}$	$\overline{17}$	$\overline{19}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{6}$	$\overline{7}$	$\overline{10}$	$\overline{12}$	$\overline{14}$	$\overline{17}$	$\overline{18}$	$\overline{0}$	$\overline{1}$
$\overline{4}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{8}$	$\overline{9}$	$\overline{12}$	$\overline{14}$	$\overline{16}$	$\overline{19}$	$\overline{0}$	$\overline{2}$	$\overline{3}$
$\overline{5}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{9}$	$\overline{10}$	$\overline{13}$	$\overline{15}$	$\overline{17}$	$\overline{0}$	$\overline{1}$	$\overline{3}$	$\overline{4}$
$\overline{8}$	$\overline{8}$	$\overline{9}$	$\overline{10}$	$\overline{12}$	$\overline{13}$	$\overline{16}$	$\overline{18}$	$\overline{0}$	$\overline{3}$	$\overline{4}$	$\overline{6}$	$\overline{7}$
$\overline{10}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{14}$	$\overline{15}$	$\overline{18}$	$\overline{0}$	$\overline{2}$	$\overline{5}$	$\overline{6}$	$\overline{8}$	$\overline{9}$
$\overline{12}$	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{16}$	$\overline{17}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{7}$	$\overline{8}$	$\overline{10}$	$\overline{11}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{5}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{15}$	$\overline{16}$	$\overline{18}$	$\overline{19}$
$\overline{15}$	$\overline{15}$	$\overline{16}$	$\overline{17}$	$\overline{19}$	$\overline{0}$	$\overline{3}$	$\overline{5}$	$\overline{7}$	$\overline{10}$	$\overline{11}$	$\overline{13}$	$\overline{14}$
$\overline{16}$	$\overline{16}$	$\overline{17}$	$\overline{18}$	$\overline{0}$	$\overline{1}$	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{11}$	$\overline{12}$	$\overline{14}$	$\overline{15}$
$\overline{18}$	$\overline{18}$	$\overline{19}$	$\overline{0}$	$\overline{2}$	$\overline{3}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	$\overline{13}$	$\overline{14}$	$\overline{16}$	$\overline{17}$
$\overline{19}$	$\overline{19}$	$\overline{0}$	$\overline{1}$	$\overline{3}$	$\overline{4}$	$\overline{7}$	$\overline{9}$	$\overline{11}$	$\overline{14}$	$\overline{15}$	$\overline{17}$	$\overline{18}$

Based on Table 2, we have:

(1) $p +_{20} q \in \overline{X}$, *for every* $p, q \in X$*;*
(2) associative property holds in $\overline{X}$*;*
(3) there exist $\overline{0} \in \overline{X}$ *such that* $p +_{20} e = e +_{20} p = p$, *for every* $p \in X$*;*
(4) every element of X *has a rough inverse element in* X *based on Table 3.*

Table 3. Inverse table on X

p	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{5}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{15}$	$\overline{16}$	$\overline{18}$	$\overline{19}$
p^{-1}	$\overline{0}$	$\overline{19}$	$\overline{18}$	$\overline{16}$	$\overline{15}$	$\overline{12}$	$\overline{10}$	$\overline{8}$	$\overline{5}$	$\overline{4}$	$\overline{2}$	$\overline{1}$

So, $(Apr(X), +_{20})$ *is a rough group in approximation space* (U, R).

Furthermore, let $Y = \{\overline{0}, \overline{1}, \overline{2}, \overline{4}, \overline{5}, \overline{8}, \overline{10}, \overline{12}, \overline{15}, \overline{16}, \overline{18}, \overline{19}\}$ *a non-empty subset of set* $U, Y \subseteq U$. *In the approximation space* (U, R), *the lower and upper approximations of* X *are obtained as follows*
$\underline{Y} = \{y | [y]_R \subseteq Y\} = \emptyset$;
$\overline{Y} = \{y | [y]_R \cap Y \neq \emptyset\} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 = U$.

Next, we define the binary operation $+_{20}$ *and* $\cdot_{20})$ *on* U. *We will show that* $Apr(Y)$ *is a rough ring.*

Table 4. Table Cayley $+_{20}$ on X

$+_{20}$	$\overline{0}$	$\overline{1}$	$\overline{6}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{14}$	$\overline{19}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{6}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{14}$	$\overline{19}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{15}$	$\overline{20}$
$\overline{6}$	$\overline{6}$	$\overline{7}$	$\overline{12}$	$\overline{15}$	$\overline{16}$	$\overline{17}$	$\overline{0}$	$\overline{5}$
$\overline{9}$	$\overline{9}$	$\overline{10}$	$\overline{15}$	$\overline{18}$	$\overline{19}$	$\overline{0}$	$\overline{3}$	$\overline{8}$
$\overline{10}$	$\overline{10}$	$\overline{11}$	$\overline{16}$	$\overline{19}$	$\overline{0}$	$\overline{1}$	$\overline{4}$	$\overline{9}$
$\overline{11}$	$\overline{11}$	$\overline{12}$	$\overline{17}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{5}$	$\overline{10}$
$\overline{14}$	$\overline{14}$	$\overline{15}$	$\overline{0}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{8}$	$\overline{13}$
$\overline{19}$	$\overline{19}$	$\overline{0}$	$\overline{5}$	$\overline{8}$	$\overline{9}$	$\overline{10}$	$\overline{13}$	$\overline{18}$

Based on Table 4, we have:

(1) $u +_{20} v \in \overline{Y}$, *for every* $u, v \in Y$*;*

(2) *commutative property holds in* Y.
(3) $(u +_{20} v) +_{20} w = u +_{20} (v +_{20} w)$, *for every* $u, v, w \in Y$;
(4) *there exists* $\overline{0} \in \overline{Y}$ *such that* $u +_{20} e = e +_{20} u = u$, *for every* $u \in Y$;
(5) *based on Table 5, every element of* Y *has a rough inverse element in* Y;

TABLE 5. Inverse table on Y

p	$\overline{0}$	$\overline{1}$	$\overline{6}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{14}$	$\overline{19}$
p^{-1}	$\overline{0}$	$\overline{19}$	$\overline{14}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{6}$	$\overline{1}$

(6) $u \cdot_{20} v \in \overline{Y}$, *for every* $u, v \in Y$;

TABLE 6. Table Cayley $\cdot_{20}$ on Y

$\cdot_{20}$	$\overline{0}$	$\overline{1}$	$\overline{6}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{14}$	$\overline{19}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{6}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{14}$	$\overline{19}$
$\overline{6}$	$\overline{0}$	$\overline{6}$	$\overline{16}$	$\overline{14}$	$\overline{0}$	$\overline{6}$	$\overline{4}$	$\overline{14}$
$\overline{9}$	$\overline{0}$	$\overline{9}$	$\overline{14}$	$\overline{1}$	$\overline{10}$	$\overline{19}$	$\overline{6}$	$\overline{11}$
$\overline{10}$	$\overline{10}$	$\overline{11}$	$\overline{16}$	$\overline{19}$	$\overline{0}$	$\overline{1}$	$\overline{4}$	$\overline{9}$
$\overline{11}$	$\overline{0}$	$\overline{10}$	$\overline{0}$	$\overline{10}$	$\overline{0}$	$\overline{10}$	$\overline{0}$	$\overline{10}$
$\overline{14}$	$\overline{0}$	$\overline{14}$	$\overline{4}$	$\overline{6}$	$\overline{0}$	$\overline{14}$	$\overline{16}$	$\overline{6}$
$\overline{19}$	$\overline{0}$	$\overline{19}$	$\overline{14}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{6}$	$\overline{1}$

Based on Table 6, we have:
(7) $(u \cdot_{20} v) \cdot_{20} w = u \cdot_{20} (v \cdot_{20} w)$, *for every* $u, v, w \in Y$;
(8) *left and right distributive law property holds in* $\overline{Y}$.

Furthermore, we will show that $Apr(X)$ *is rough module. We define a mapping* $\overline{Y} \times \overline{X} \to \overline{X}, (u, p) \to u \cdot_{20} p$, *for every* $u \in Apr(Y)$ *and* $p \in Apr(X)$.

Based on Table 7, we have:

(1) $u \cdot_{20} (p +_{20} q) = (u \cdot_{20} p) +_{20} (u \cdot_{20} q)$, *for every* $u \in Apr(Y)$ *and* $p, q \in Apr(X)$;

TABLE 7. Table scalar multiplication $\cdot_{20}$

$\cdot_{20}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{5}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{15}$	$\overline{16}$	$\overline{18}$	$\overline{19}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{5}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{15}$	$\overline{16}$	$\overline{18}$	$\overline{19}$
$\overline{6}$	$\overline{0}$	$\overline{6}$	$\overline{12}$	$\overline{4}$	$\overline{10}$	$\overline{8}$	$\overline{0}$	$\overline{12}$	$\overline{10}$	$\overline{16}$	$\overline{8}$	$\overline{14}$
$\overline{9}$	$\overline{0}$	$\overline{9}$	$\overline{18}$	$\overline{16}$	$\overline{5}$	$\overline{12}$	$\overline{10}$	$\overline{8}$	$\overline{15}$	$\overline{4}$	$\overline{2}$	$\overline{11}$
$\overline{10}$	$\overline{0}$	$\overline{10}$	$\overline{0}$	$\overline{10}$	$\overline{0}$	$\overline{10}$	$\overline{0}$	$\overline{10}$	$\overline{0}$	$\overline{10}$	$\overline{0}$	$\overline{10}$
$\overline{11}$	$\overline{0}$	$\overline{11}$	$\overline{2}$	$\overline{4}$	$\overline{15}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{5}$	$\overline{16}$	$\overline{18}$	$\overline{9}$
$\overline{14}$	$\overline{0}$	$\overline{14}$	$\overline{8}$	$\overline{16}$	$\overline{10}$	$\overline{12}$	$\overline{0}$	$\overline{8}$	$\overline{10}$	$\overline{4}$	$\overline{12}$	$\overline{6}$
$\overline{19}$	$\overline{0}$	$\overline{19}$	$\overline{18}$	$\overline{16}$	$\overline{15}$	$\overline{12}$	$\overline{10}$	$\overline{8}$	$\overline{5}$	$\overline{4}$	$\overline{2}$	$\overline{1}$

(2) $(u +_{20} v) \cdot_{20} p = (u \cdot_{20} p) +_{20} (v \cdot_{20} p)$, *for every* $u, v \in Apr(Y)$ *and* $p \in Apr(X)$;
(3) $(u \cdot_{20} v) \cdot_{20} p = u \cdot_{20} (v \cdot_{20} p)$, *for every* $u, v \in Apr(Y)$ *and* $p \in Apr(X)$;
(4) *there is a rough identity element* $\overline{1} \in Apr(Y)$, *such that* $\overline{1} \cdot_{20} p = p$, *for every* $p \in Apr(X)$.
So, it proves that $Apr(X)$ *is a rough module over the rough ring* $Apr(Y)$.

Now, we investigate the properties of the rough module over a rough ring.

Let S be submodule of M, for $a, b \in M$ we say a is congruent to b mod S, written as $a \equiv b \pmod{S}$ if $a - b \in S$. Therefore, a submodule S of M does induce an equivalence relation on M and the corresponding set M/S of equivalence classes in $M/S = a + S | a \in M$.

Proposition 3.3. *Let S be submodule of M, for $a, b \in M$ obtained a congruent relation modulo S, written as $a \equiv b$ (mod S) is an equivalence relation.*

Proof. We will proof that congruent relation modulo S is an equivalence relation.

(1) For $a \in M$, then $a - a = 0 \in S$ then $a \equiv a \pmod{S}$. So, this relation is reflexive.
(2) Given any $a, b \in M$ with $a \equiv b \pmod{S}$. We will show that $b \equiv a \pmod{S}$. Since $a \equiv b \pmod{S}$, consequently $a - b \in S$. Therefore, it obtained $-(a - b) = b - a \in S$, that it implies $b \equiv a \pmod{S}$. So, this relation is symmetric.
(3) Given any $a, b, c \in M$, with $a \equiv b \pmod{S}$ and $b \equiv c \pmod{S}$ we will show that $a \equiv c \pmod{S}$. Since $a \equiv b \pmod{S}$ then $a - b \in S$, and $b \in c \pmod{S}$ then $b - c \in S$. Therefore, it obtained $a - b + (b - c) = a - c \in S$ that it implies $a \in c \pmod{S}$. So, this relation is transitive.

It proves that a congruent relation modulo S is an equivalence relation. □

Definition 3.4. *Let S be a submodule of M and A a non-empty subset of M, then the sets $\underline{A} = x \in M | x + S \in A$ and $\overline{A} = x \in M | (x + S) \cap A \neq \emptyset$ are called lower and upper approximation of the set A with respect to the submodule S. In this case, the pair (M, S) as approximation space* [24].

Definition 3.5. *Let M be an R-module, S be a submodule of M and $Apr(S) = (\overline{S}, \underline{S})$ is a rough set in the approximation space (M, S). If $\overline{S}$ and $\underline{S}$ are submodules of M, then we call $Apr(S)$ a rough module* [24].

Example 3.6. *Let $M = \mathbb{Z}_{24} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, .., \overline{23}\}$ be $\mathbb{Z}$-module, and $S = \{\overline{0}, \overline{8}, \overline{16}\}$ be a submodule of M. Based on Proposition 3.3, the congruent relation modulo S can be used as an equivalence relation written as $a \equiv b$ (mod S) for $a, b \in M$ and form an approximation space (M, S). We have eight equivalence classes from left coset of submodule S on M.*

TABLE 8. The equivalence classes of $\mathbb{Z}_{24}$

The Equivalence Classes	*The Element*
$E_1 = \overline{0} + S$	$\{\overline{0}, \overline{8}, \overline{16}\}$
$E_2 = \overline{1} + S$	$\{\overline{1}, \overline{9}, \overline{17}\}$
$E_3 = \overline{2} + S$	$\{\overline{2}, \overline{10}, \overline{18}\}$
$E_4 = \overline{3} + S$	$\{\overline{3}, \overline{11}, \overline{19}\}$
$E_5 = \overline{4} + S$	$\{\overline{4}, \overline{12}, \overline{20}\}$
$E_6 = \overline{5} + S$	$\{\overline{5}, \overline{13}, \overline{21}\}$
$E_7 = \overline{6} + S$	$\{\overline{6}, \overline{14}, \overline{22}\}$
$E_8 = \overline{7} + S$	$\{\overline{7}, \overline{15}, \overline{23}\}$

Given $A \subseteq M$, with $A = \{\overline{0}, \overline{2}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}, \overline{22}\}$. Then the lower and upper approximation are of A are $\underline{A} = E_1 \cup E_5 = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}\}$ and $\overline{A} = E_1 \cup E_3 \cup E_5 \cup E_7 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\}$.
We will show that $\overline{A}$ is submodule of M.

(1) For every $a, b \in \overline{A}$, we have $a -_{24} b \in \overline{A}$.
(2) For every $r \in Z, a \in \overline{A}$, we have $r \cdot_{24} a \in \overline{A}$.

It proves that $\overline{A}$ is submodule of M.

Next, we will show that $\overline{A}$ is submodule of M.

(1) *For every $a, b \in \underline{A}$, we have $a -_{24} b \in \underline{A}$.*
(2) *For every $r \in Z, a \in]\underline{A}$, we have $r \cdot_{24} a \in \underline{A}$.*

It proves that $\underline{A}$ is submodule of M.
Since $\overline{A}$ and $\underline{A}$ are submodules of M, then $Apr(A)$ is a rough module over ring Z.

Proposition 3.7. *Let A and B be two submodules of M, then $\overline{B}$ is a submodule of M* [24].

Proof. Suppose $a, b \in \overline{B}$ and $r \in R$, then $(a + A) \cap B \neq \emptyset$ and $(b + A) \cap B \neq \emptyset$. So there exists $x \in (a + A) \cap B \neq \emptyset$ and $y \in (b + A) \cap B \neq \emptyset$. Since B is a submodule of M, we get $x - y \in B$ and $x - y \in (a + A) - (b + A) = a - b + A$, Hence $(a - b + A) \cap B \neq \emptyset$, which implies that $(a-b) \in \overline{B}$. Also, we have $rx \in B$ and $r(a+A) = ra+rA \subset ra+A$ which implies $(ra + A) \cap B \neq \emptyset$ and $ra \in \overline{B}$. Therefore, $\overline{B}$ is a submodule of M. □

Before we construct the rough submodule, we recall the definition of the rough submodule as follows.

Definition 3.8. *A rough subset $Apr(N) \neq \emptyset$ of a rough module $Apr(M)$ is called a rough submodule of $Apr(M)$, if $Apr(N)$ satisfies the following:*

(1) *$Apr(N)$ is a rough subgroup of $Apr(M)$;*
(2) *$ay \in \overline{N}, a \in R, y \in N$.*

[11].

Based on Example 3.2, we have $Apr(X)$ is a rough module over rough ring $Apr(Y)$. Then we can construct the rough submodule on $Apr(X)$.

Example 3.9. *Let $N \subseteq X$, with $N = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}\}$. Based on equivalence classes on approximation space (U, R), we get the lower and upper approximation of $N \cdot \underline{N} = a|[a]_R \subseteq N = \emptyset$ and*
$\overline{N} = \{a|[a]_R \cap N \neq \emptyset\} = E_1 \cup E_3 \cup E_5 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}\}$.

Furthermore, we will show that $Apr(N)$ is a rough submodule of $Apr(M)$.

(1) *For every $a, b \in Apr(N)$, we have $a -_{20} b \in \overline{N}$.*

TABLE 9. Table Cayley $-_{20}$ on N

$-_{20}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{12}$	$\overline{16}$
$\overline{0}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{12}$	$\overline{16}$
$\overline{4}$	$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{12}$
$\overline{8}$	$\overline{8}$	$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{8}$
$\overline{12}$	$\overline{12}$	$\overline{8}$	$\overline{4}$	$\overline{0}$	$\overline{4}$
$\overline{16}$	$\overline{16}$	$\overline{12}$	$\overline{8}$	$\overline{4}$	$\overline{0}$

(2) *For every $u \in Apr(Y)$, $a \in Apr(N)$, we have $u \cdot_{20} a \in \overline{N}$.*

TABLE 10. Table scalar multiplication $\cdot_{20}$

$\cdot_{20}$	$\overline{0}$	$\overline{1}$	$\overline{6}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{14}$	$\overline{19}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{4}$	$\overline{16}$	$\overline{0}$	$\overline{4}$	$\overline{16}$	$\overline{16}$
$\overline{8}$	$\overline{0}$	$\overline{8}$	$\overline{8}$	$\overline{12}$	$\overline{0}$	$\overline{8}$	$\overline{12}$	$\overline{12}$
$\overline{12}$	$\overline{0}$	$\overline{12}$	$\overline{12}$	$\overline{8}$	$\overline{0}$	$\overline{12}$	$\overline{8}$	$\overline{8}$
$\overline{16}$	$\overline{0}$	$\overline{16}$	$\overline{16}$	$\overline{4}$	$\overline{0}$	$\overline{16}$	$\overline{4}$	$\overline{4}$

It proves that $Apr(N)$ is rough submodule of rough module $Apr(X)$ over rough ring $Apr(Y)$.

Proposition 3.10. *Every rough subgroup $Apr(H)$ of a rough commutative group $Apr(G)$ is a rough submodule of rough $\mathbb{Z}$-module $Apr(G)$, for $n \cdot x = x^n \in \overline{H}, n \in \mathbb{Z}, x \in H$* [11].

Proof. Since the requirement to be a rough submodule is that a rough set is a rough subgroup of a rough commutative group $Apr(H)$ satisfies the first condition. Next, given a ring $\mathbb{Z}$ with $n \in \mathbb{Z}$ and a set H with $x \in H$ to satisfy the second condition with the operation $n \cdot x = x^n \in \overline{H}$. Since $x \in H$ so $x \in \overline{H}$ and if operated using the same operation on the $Apr(G)$ module with $n \in \mathbb{Z}$ then $x^n \in \overline{H}$. Therefore, we get that $Apr(H)$ is a rough submodule of the $\mathbb{Z}$-module $Apr(G)$. □

Definition 3.11. *Let $Apr(A) = (\underline{A}, \overline{A})$ and $Apr(B) = (\underline{B}, \overline{B})$ be any two rough sets in the approximation space (U, R). Then*

(1) $Apr(A) \cup Apr(B) = (\underline{A} \cup \underline{B}, \overline{A} \cup \overline{B})$.
(2) $Apr(A) \cap Apr(B) = (\underline{A} \cap \underline{B}, \overline{A} \cap \overline{B})$.

[25].

Definition 3.12. *For every approximation space (U, R) and every subset $A, B \subseteq U$. With $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then*

(1) $\underline{(A \cap B)} = \underline{A} \cap \underline{B}$.
(2) $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$.

[25].

Therefore, we can define that $(A \cap B) = \underline{A \cap B}\overline{A \cap B}$. and $Apr(A) \cap Apr(B) = (\underline{A} \cap \underline{B}, \overline{A} \cap \overline{B})$. Since $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$, we have $Apr(A \cap B) \subseteq Apr(A) \cap Apr(B)$.

Proposition 3.13. *Let A and B be two submodules of R-module M. If $Apr(A)$ and $Apr(B)$ are rough submodule of rough module $Apr(M)$ over rough ring $Apr(R)$ with $\overline{A} \cap \overline{B} = U$, then $Apr(A \cap B)$ is rough submodule of rough module $Apr(M)$.*

Proof.

(1) Given any $x, y \in Apr(A \cap B)$, then $x, y \in \overline{(A \cap B)} \subseteq \overline{A}$ and $x, y \in \overline{(A \cap B)} \subseteq \overline{B}$. Since $Apr(A)$ and $Apr(B)$ are rough submodule, we get $x - y \in \overline{A}$ and $x - y \in \overline{B}$. Therefore $x - y \in \overline{(A \cap B)}$, for $x, y \in Apr(A \cap B)$.
(2) Given any $r \in Apr(R), x \in Apr(A \cap B)$. Since $x \in \overline{A}$ and $x \in \overline{B}$, we get $rx \in \overline{A}$ and $rx \in \overline{B}$. Therefore $rx \in \overline{A \cap B}$.

It proves that $Apr(A \cap B)$ is rough submodule of rough module $Apr(M)$. □

Example 3.14. *Let $U = \mathbb{Z}_{24} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \ldots, \overline{23}\}$. We define aRb if only if $a - b = 5z$ for $z \in \mathbb{Z}$. We have five equivalence classes in the following table based on the equivalence relation.*

Table 11. The equivalence classes of $\mathbb{Z}_{20}$

The Equivalence Classes	*The Element*
E_1	$\{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}\}$
E_2	$\{\overline{1}, \overline{6}, \overline{11}, \overline{16}, \overline{21}\}$
E_3	$\{\overline{2}, \overline{7}, \overline{12}, \overline{17}, \overline{22}\}$
E_4	$\{\overline{3}, \overline{8}, \overline{13}, \overline{18}, \overline{23}\}$
E_5	$\{\overline{4}, \overline{9}, \overline{14}, \overline{19}\}$

Furthermore, let $X = \{\overline{0}, \overline{1}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}, \overline{12}, \overline{15}, \overline{16}, \overline{18}, \overline{20}, \overline{21}, \overline{23}\}$ a non-empty subset of set $U, X \subseteq U$. In the approximation space (U, R), the lower and upper approximations of X are obtained as follows:
$\underline{X} = \{x | [x]_R \subseteq X\} = \emptyset$

$\overline{X} = \{x|[x]_R \cap X \neq \emptyset\} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 = U$.

Next, we define the binary operation $(Apr(X), +_{24})$. We will show that $Apr(X)$ is rough group.

(1) For every $p, q \in X$, we have $p +_{24} q \in \overline{X}$.
(2) Associative property holds in $\overline{X}$.
(3) There exist $\overline{0} \in \overline{X}$ such that $p +_{20} e = e +_{24} p = p$, for every $p \in X$.
(4) Every element of X has a rough inverse element in X.

Table 12. Inverse table on X

p	$\overline{0}$	$\overline{1}$	$\overline{3}$	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{9}$	$\overline{12}$	$\overline{15}$	$\overline{16}$	$\overline{18}$	$\overline{20}$	$\overline{21}$	$\overline{23}$
p^{-1}	$\overline{0}$	$\overline{23}$	$\overline{21}$	$\overline{20}$	$\overline{18}$	$\overline{16}$	$\overline{15}$	$\overline{12}$	$\overline{9}$	$\overline{8}$	$\overline{6}$	$\overline{4}$	$\overline{3}$	$\overline{1}$

So, $(Apr(X), +_{24})$ is a rough group in approximation space (U, R).

Furthermore, let $Y = \{\overline{0}, \overline{1}, \overline{2}, \overline{7}, \overline{10}, \overline{14}, \overline{17}, \overline{22}, \overline{23}\}$ a non-empty subset of set U, $Y \subseteq U$. In the approximation space (U, R), the lower and upper approximations of X are obtained as follows:
$\underline{Y} = \{y|[y]_R \subseteq Y\} = \emptyset$
$\overline{Y} = \{y|[y]_R \cap Y \neq \emptyset\} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 = U$.

Next, we define the binary operation $(Apr(Y), +_{24}, \cdot_{24})$. We will show that $Apr(Y)$ is arough ring.

(1) For every $u, v \in Y$, we have $u +_{24} v \in \overline{Y}$.
(2) Commutative property holds in Y.
(3) Assosiative property holds in $\overline{Y}$.
(4) There exist $\overline{0} \in \overline{Y}$ such that $u +_{24} e = e +_{24} u = u$, for every $u \in Y$.
(5) Every element of Y has a rough inverse element in Y.

Table 13. Inverse table on Y

u	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{10}$	$\overline{14}$	$\overline{17}$	$\overline{22}$	$\overline{23}$
u^{-1}	$\overline{0}$	$\overline{23}$	$\overline{22}$	$\overline{17}$	$\overline{14}$	$\overline{10}$	$\overline{7}$	$\overline{2}$	$\overline{1}$

(6) For $u, v \in Y$ then $u \cdot_{24} v \in \overline{Y}$.

Table 14. Table Cayley $\cdot_{24}$ on Y

$\cdot_{24}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{10}$	$\overline{14}$	$\overline{17}$	$\overline{22}$	$\overline{23}$
0	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
1	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{10}$	$\overline{14}$	$\overline{17}$	$\overline{22}$	$\overline{23}$
2	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{14}$	$\overline{20}$	$\overline{4}$	$\overline{10}$	$\overline{20}$	$\overline{22}$
7	$\overline{0}$	$\overline{7}$	$\overline{14}$	$\overline{1}$	$\overline{22}$	$\overline{2}$	$\overline{23}$	$\overline{10}$	$\overline{17}$
10	$\overline{0}$	$\overline{10}$	$\overline{20}$	$\overline{22}$	$\overline{4}$	$\overline{20}$	$\overline{2}$	$\overline{4}$	$\overline{14}$
14	$\overline{0}$	$\overline{14}$	$\overline{4}$	$\overline{2}$	$\overline{20}$	$\overline{4}$	$\overline{22}$	$\overline{20}$	$\overline{10}$
17	$\overline{0}$	$\overline{17}$	$\overline{10}$	$\overline{23}$	$\overline{2}$	$\overline{22}$	$\overline{1}$	$\overline{14}$	$\overline{7}$
22	$\overline{0}$	$\overline{22}$	$\overline{20}$	$\overline{10}$	$\overline{4}$	$\overline{20}$	$\overline{14}$	$\overline{4}$	$\overline{2}$
23	$\overline{0}$	$\overline{23}$	$\overline{22}$	$\overline{17}$	$\overline{14}$	$\overline{10}$	$\overline{7}$	$\overline{2}$	$\overline{1}$

(7) For every $u, v, w \in Y$, we have $(u \cdot_{24} v) \cdot_{24} w = u \cdot_{24} (v \cdot_{24} w)$, so assosiative property holds in $\overline{Y}$.
(8) Left and right distributive law property holds binary operation $\cdot_{24}$ in $\overline{Y}$.

Furthermore, let A subset of set X with $A = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}\} \subseteq X$. Based on equivalence classes on approximation space (U, R). We get the lower and upper approximations of A are obtained as follows:
$\underline{A} = \{a|[a]_R \subseteq A\} = \emptyset$
$\overline{A} = \{a|[a]_R \cap A \neq \emptyset\} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 = U$.

We will show that $Apr(A)$ is a rough submodule of $Apr(X)$, if $Apr(A)$ satisfies the following:

(1) For every $a, b \in Apr(A)$, we have $a -_{24} b \in \overline{A}$.

Table 15. Table Cayley $-_{24}$ on Y

$-_{24}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{12}$	$\overline{16}$	$\overline{20}$
0	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{12}$	$\overline{16}$	$\overline{20}$
4	$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{12}$	$\overline{16}$
8	$\overline{8}$	$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{12}$
12	$\overline{12}$	$\overline{8}$	$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{8}$
16	$\overline{16}$	$\overline{12}$	$\overline{8}$	$\overline{4}$	$\overline{0}$	$\overline{4}$
20	$\overline{20}$	$\overline{16}$	$\overline{12}$	$\overline{8}$	$\overline{4}$	$\overline{0}$

(2) For every $u \in Apr(Y)$, $a \in Apr(A)$, we have $u \cdot_{24} a \in \overline{A}$.

Table 16. Table scalar multiplication $\cdot_{24}$

$\cdot_{24}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{10}$	$\overline{14}$	$\overline{17}$	$\overline{22}$	$\overline{23}$
0	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
4	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{4}$	$\overline{16}$	$\overline{8}$	$\overline{20}$	$\overline{16}$	$\overline{20}$
8	$\overline{0}$	$\overline{8}$	$\overline{16}$	$\overline{8}$	$\overline{8}$	$\overline{16}$	$\overline{16}$	$\overline{8}$	$\overline{16}$
12	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{12}$
16	$\overline{0}$	$\overline{16}$	$\overline{8}$	$\overline{16}$	$\overline{16}$	$\overline{8}$	$\overline{8}$	$\overline{16}$	$\overline{8}$
20	$\overline{0}$	$\overline{20}$	$\overline{16}$	$\overline{20}$	$\overline{8}$	$\overline{16}$	$\overline{4}$	$\overline{8}$	$\overline{4}$

It proves that $Apr(A)$ is rough submodule of rough module $Apr(X)$ over rough ring $Apr(Y)$.

Furthermore, let B subset of set X with $B = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}\} \subseteq X$ Based on equivalence classes on approximation space (U, R). We get the lower and upper approximations of B are obtained as follows:
$\underline{B} = \{b|[b]_R \subseteq B\} = \emptyset$
$\overline{B} = \{b|[b]_R \cap B \neq \emptyset\} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 = U$.

We will show that $Apr(B)$ is a rough submodule of $Apr(X)$, if $Apr(B)$ satisfies the following:

(1) For every $a, b \in Apr(B)$, we have $a -_{24} b \in \overline{B}$.

Table 17. Table Cayley $\cdot_{24}$ on B

$\cdot_{24}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{9}$	$\overline{12}$	$\overline{15}$	$\overline{18}$	$\overline{21}$
0	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{9}$	$\overline{12}$	$\overline{15}$	$\overline{18}$	$\overline{21}$
3	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{9}$	$\overline{12}$	$\overline{15}$	$\overline{18}$
6	$\overline{6}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{9}$	$\overline{12}$	$\overline{15}$
9	$\overline{9}$	$\overline{6}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{9}$	$\overline{12}$
12	$\overline{12}$	$\overline{9}$	$\overline{6}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{9}$
15	$\overline{15}$	$\overline{12}$	$\overline{9}$	$\overline{6}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$
18	$\overline{18}$	$\overline{15}$	$\overline{12}$	$\overline{9}$	$\overline{6}$	$\overline{3}$	$\overline{0}$	$\overline{3}$
21	$\overline{21}$	$\overline{18}$	$\overline{15}$	$\overline{12}$	$\overline{9}$	$\overline{6}$	$\overline{3}$	$\overline{0}$

(2) For every $u \in Apr(Y), b \in Apr(B)$, we have $u \cdot_{24} b \in \overline{B}$.

Table 18. Table scalar multiplication $\cdot_{24}$

$\cdot_{24}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{10}$	$\overline{14}$	$\overline{17}$	$\overline{22}$	$\overline{23}$
0	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
3	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{21}$	$\overline{6}$	$\overline{18}$	$\overline{3}$	$\overline{18}$	$\overline{21}$
6	$\overline{0}$	$\overline{6}$	$\overline{12}$	$\overline{18}$	$\overline{12}$	$\overline{12}$	$\overline{6}$	$\overline{12}$	$\overline{18}$
9	$\overline{0}$	$\overline{9}$	$\overline{18}$	$\overline{15}$	$\overline{18}$	$\overline{6}$	$\overline{9}$	$\overline{6}$	$\overline{15}$
12	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{12}$
15	$\overline{0}$	$\overline{15}$	$\overline{6}$	$\overline{9}$	$\overline{6}$	$\overline{18}$	$\overline{15}$	$\overline{18}$	$\overline{9}$
18	$\overline{0}$	$\overline{18}$	$\overline{12}$	$\overline{6}$	$\overline{12}$	$\overline{12}$	$\overline{18}$	$\overline{12}$	$\overline{6}$
21	$\overline{0}$	$\overline{12}$	$\overline{18}$	$\overline{3}$	$\overline{18}$	$\overline{6}$	$\overline{9}$	$\overline{6}$	$\overline{3}$

(3) It proves that $Apr(B)$ is rough submodule of rough module $Apr(X)$ over rough ring $Apr(Y)$.

Next, since $Apr(A), Apr(B) \subseteq Apr(X)$ are rough submodule. We will show that $Apr(A \cap B)$ is rough submodule of rough module $Apr(X).A = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}\}$ and $B = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}\}$ we get $A \cap B = \{\overline{0}, \overline{12}\}$. Based on equivalence classes on approximation space (U, R). We get the lower and upper approximations of $A \cap B$ are obtained as follows: $\underline{(A \cap B)} = \{a | [a]_R \subseteq (A \cap B)\} = \underline{A} \cap \underline{B} = \emptyset$, and
$\overline{(A \cap B)} = \{a | [a]_R \cap (A \cap B) \emptyset\} = E_1 \cup E_3 = \{\overline{0}, \overline{2}, \overline{5}, \overline{7}, \overline{10}, \overline{12}, \overline{15}, \overline{17}, \overline{20}, \overline{22}\} \subseteq \overline{A} \cap \overline{B}$.

We will show that $Apr(A \cap B)$ is rough submodule of rough module $Apr(X)$ over rough ring $Apr(Y)$.

(1) For every $a, b \in Apr(A \cap B)$, we have $a -_{24} b \in \overline{(A \cap B)}$.

Table 19. Table Cayley $-_{24}$ on $A \cap B$

$-_{24}$	$\overline{0}$	$\overline{12}$
0	$\overline{0}$	$\overline{12}$
12	$\overline{12}$	$\overline{0}$

(2) For every $u \in Apr(Y)$, $a \in Apr(A \cap B)$, we have $u \cdot_{24} a \in \overline{(A \cap B)}$.

Table 20. Table scalar multiplication $\cdot_{24}$

$\cdot_{24}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{10}$	$\overline{14}$	$\overline{17}$	$\overline{22}$	$\overline{23}$
0	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
12	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{0}$	$\overline{12}$	$\overline{0}$	$\overline{12}$

It proves that $Apr(A \cap B)$ is rough submodule of rough module $Apr(X)$ over rough ring $Apr(Y)$.

Proposition 3.15. *Let $Apr(S_\alpha)$ for every $\alpha \in \triangle$ is rough submodule of rough module $Apr(M)$ over rough ring $Apr(R)$. If $Apr(S_\alpha)$ for every $\alpha \in \triangle$ is rough submodule with $\cap_{\alpha \in \triangle} \overline{S_\alpha} = \overline{\cap_{\alpha \in \triangle} S_\alpha}$. Then, $\cap_{\alpha \in \triangle} Apr(S_\alpha)$ is rough submodule of rough module $Apr(M)$ over rough ring $Apr(R)$.*

Proof.

(1) Given any $x, y \in \cap_{\alpha \in \triangle} Apr(S_\alpha)$ then $x, y \in Apr(S_\alpha)$ for every $\alpha \in \triangle$. Since $Apr(S_\alpha)$ for every $\alpha \in \triangle$ is rough submodule, we get $x - y \in \overline{(S_\alpha)}$ for every $\alpha \in \triangle$. That implies $x - y \in \triangle_{\alpha \in \triangle} \overline{S_\alpha} = \overline{\cap_{\alpha \in \triangle}} (S_\alpha)$.

(2) Given any $r \in Apr(R)$, $x \in \cap_{\alpha\in\triangle} Apr(S_\alpha)$. Since $x \in \cap_{\alpha\in\triangle)} Apr(S_\alpha)$ then $x \in Apr(S_\alpha)$ for every $\alpha \in \triangle$. Since $x \in Apr(S_\alpha)$ for every $\alpha \in \triangle$ is rough submodule, we get $rx \in \overline{S_\alpha}$ for every $\alpha \in \triangle$. That implies $rx \in \cap_{\alpha\in\triangle}\overline{(S_\alpha)} = \overline{\cap_{\alpha\in\triangle}(S_\alpha)}$.

It proves that $\cap_{\alpha\in\triangle} Apr(S_\alpha)$ is rough submodule of rough module $Apr(M)$. □

4. Conclusion

A rough module over ring rough can be constructed using approximation space (U, R) with U be universal set and R be equivalence relation, and an approximation space (M, S) with M be module and S be submodule. Regarding the properties of the rough submodule, we get that if $Apr(A), Apr(B)$ is a rough submodule of the rough module $Apr(M)$ over a ring rough $Apr(R)$ with $\overline{A} \cap \overline{B} = U$, then $Apr(A \cap B)$ is also a rough submodule of the rough module $Apr(M)$. Furthermore, If $Apr(S_\alpha)$for every $\alpha \in \triangle$ is a rough submodule with $\cap_{\alpha\in\triangle}\overline{S_\alpha} = \overline{\cap_{\alpha\in\triangle}(S_\alpha)}$, then $\cap_{\alpha\in\triangle} Apr(S_\alpha)$ is also a rough submodule of the rough module $Apr(M)$ over a ring rough $Apr(R)$.

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