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Discrete Dynamical System Generated by Set-valued Function in Metric Spaces

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Abstract
The main purpose of this article is to study the behavior of the solutions of discrete dynamical systems generated by set-valued mappings within metric spaces. Specifically, it investigates the convergence of iterations in the presence of computational errors for such mappings. The approach employed involves orbit class set operations. Furthermore, the study establishes a necessary condition for the existence and uniqueness of fixed points for set-valued mappings.

Keywords: *Set-valued , Orbit systems, fixed point.*

1. Introduction

Discrete dynamical systems with a generating function of a single-valued function are widely studied and commonly appear in the literature. In contrast, systems generated by set-valued functions receive significantly less attention and are seldom explored [1]. This article aims to delve into this less-charted area, particularly due to its relevance in addressing various problems, including economic models based on set-valued mappings [2]. In pure exchange economics, for example, the demand function is typically represented as a set-valued mapping. This highlights the importance of studying set-valued dynamical systems, as understanding these systems could contribute to identifying equilibrium points in economic systems.

The study of discrete dynamical systems revolves around understanding the behavior of their orbits. This can be achieved through two approaches: first, by analyzing the behavior of different initial values while keeping the parameter in a fixed amount, and second, by studying how this behavior changes when the parameters are varied. The first method is known as initial value analysis, while the second is referred to as the parameter space analysis [3].

The study of the convergence of iterations for contractive type mappings has been an important topic in Nonlinear Functional Analysis since Banach's seminal work [4] on the existence of a unique fixed point for a strict contraction [5]. Banach's renowned theorem not only guarantees the existence of a unique fixed point but also ensures the convergence of iterates to this fixed point. However, when it comes to set-valued mappings, the analysis becomes more complex and less understood, leading to intriguing and challenging results. In the case of set-valued mappings, convergence of all trajectories in the dynamical system induced by the mapping is not guaranteed. Instead, convergent trajectories must be specially constructed

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or designed to ensure their convergence, as the general behavior of the system can be more complex and unpredictable [6].

This paper investigates the convergence of iterations generated by set-valued mappings. Generally, the orbits resulting from the iterations of systems generated by set-valued mappings are not unique. Therefore, this paper aims to analyze the system in a way that identifies the orbit that converges to the fixed point.

2. Preliminary

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric space and $\mathcal{P}_o(\mathcal{Y})$ the family of all non-empty subsets of $\mathcal{Y}$. The mapping $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{P}_o(\mathcal{Y})$ is called set-valued maps. The graph of a set-valued maps F from $\mathcal{X}$ to $\mathcal{P}_o(\mathcal{Y})$ is defined

$$Graph(\mathcal{F}) = \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid y \in \mathcal{F}(x)\}$$

If $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{P}_o(\mathcal{X})$ is set-valued maps. A point x is called a fixed point of $\mathcal{F}$ if it is a solution of the inclusion

$$x \in \mathcal{F}(x)$$

Set-valued mappings are frequently encountered in fields such as control theory, economics, and other disciplines. In economic exchange, for example, the demand function is a set-valued mapping. Consumers (or individuals) can choose from a set of commodity bundles based on their preferences. Consequently, the demand function is represented as a set-valued mapping, as it captures the multiple options available to consumers, rather than just a single choice.[7]

For set-valued $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{P}_o(\mathcal{Y})$, upper inverse of $\mathcal{F}$ is defined as the set

$$\mathcal{F}^+(B) = \{x \in \mathcal{X} \mid \mathcal{F}(x) \cap B \neq \emptyset, \quad B \subset \mathcal{Y}\}$$

whereas a lower inverse of $\mathcal{F}$ is

$$\mathcal{F}^-(B) = \{x \in \mathcal{X} \mid \mathcal{F}(x) \subset B, \quad B \subset \mathcal{Y}\}$$

Definition 2.1. [8] *A map $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{P}_o(\mathcal{Y})$ is called upper semi-continuous (u.s.c) on $\mathcal{X}$ if for every open subset G of $\mathcal{Y}$, $\mathcal{F}^+(G)$ is open in $\mathcal{X}$. It is called lower semi-continuous on $\mathcal{X}$ if for every open subset G of $\mathcal{Y}$, $\mathcal{F}^-(G)$ is an open set in $\mathcal{X}$. $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{P}_o(\mathcal{Y})$ is called continuous on $\mathcal{X}$ if $\mathcal{F}$ upper and lower semi-continuous (l.s.c) on $\mathcal{X}$.*

We are thus led to introduce the class of functions $f : \mathcal{X} \longrightarrow [0, +\infty]$ and to associate them with their domain

$$Dom(f) = \{x \in \mathcal{X} \mid f(x) < +\infty\} \tag{1}$$

Definition 2.2. *A function $f : \mathcal{X} \longrightarrow [0, +\infty]$ is **strict** if $Dom(f) \neq \emptyset$*

The continuity of the function f would be reviewed through the upper and lower semi-continuous, as defined below.

Definition 2.3. *A functions $f : \mathcal{X} \longrightarrow [0, +\infty]$ is called l.s.c at $x_0 \in \mathcal{X}$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$f(x_0) - \epsilon \leq f(x) \tag{2}$$

for every $x \in B(x_0, \delta)$.

A function f is u.p.s if $(-f)$ is l.s.c and f is continuous if it is l.s.c and u.p.s

This theorem forms the basis for the fixed point existence in set-valued mappings, as introduced by Caristi.

Theorem 2.4 (Ekeland [9]). *Suppose that $\mathcal{X}$ is a complete metric space and that $f : \mathcal{X} \longrightarrow [0, +\infty]$ is strict and l.s.c. Consider $\varepsilon > 0$ and $x_0 \in Dom(f)$. There exist $\bar{x} \in \mathcal{X}$ such that*

(i) $f(x_0) \geq f(\bar{x}) + \varepsilon d(x_0, \bar{x})$
(ii) $f(\bar{x}) < f(x) + \epsilon d(x, \bar{x})$, for all $x \neq \bar{x}$,

Proof. To simplify, we may take $\varepsilon = 1$. We shall associate the function f with the set-valued map $\mathcal{F} : \mathcal{X} \longleftrightarrow \mathcal{P}_o(\mathcal{X})$ defined by

$$\mathcal{F}(x) = \{y \in \mathcal{X} \mid f(x) \geq f(y) + d(y, x)\} \tag{3}$$

for all $x \in \mathcal{X}$. The sets $\mathcal{F}(x)$ is closed and $\mathcal{F}(x)$ has the following property

(a) $y \in \mathcal{F}(y)$
(b) if $y \in \mathcal{F}(x)$ then $\mathcal{F}(y) \subset \mathcal{F}(x)$.

We know that $f(x) < \infty$. If $y \in \mathcal{F}(x)$ and $z \in \mathcal{F}(y)$, then we obtain the inequality

$$f(y) \geq f(z) + d(y, z) \tag{4}$$

and

$$f(x) \geq f(y) + d(x, z). \tag{5}$$

By using the triangle inequality, we obtain as follows

$$f(x) \geq f(z) + d(x, z) \tag{6}$$

which implies that $z \in F(x)$. The part (i) is proven.
Next, part (ii) will be proven. We define the function g on Dom(f) as follows

$$g(y) = \inf_{z \in \mathcal{F}(y)} f(z). \tag{7}$$

For every $y \in \mathcal{F}(x)$, we have $d(x, y) \leq f(x) - g(x)$, which implies that diameter of $\mathcal{F}(x)$ is bounded,

$$Diam(\mathcal{F}(x)) \leq 2(f(x) - g(x)) < \infty. \tag{8}$$

Next, we define the sequence x_{n+1} in $\mathcal{F}(x_n)$ such that $f(x_{n+1}) \leq g(x_n) + 2^{-n}$ which beginning with x_0. Since $\mathcal{F}(x_{n+1}) \subset \mathcal{F}(x_n)$, by property (b) of $\mathcal{F}$, we have

$$g(x_n) \leq g(x_{n+1}) \tag{9}$$

Since we always have $g(y) \leq f(y)$, we obtained the inequalities

$$0 \leq f(x_{n+1}) \leq g(x_n) + 2^{-n} \leq g(x_{n+1}) + 2^{-n} \tag{10}$$

or

$$0 \leq f(x_{n+1}) - g(x_{n+1}) \leq 2^{-n}. \tag{11}$$

Consequently, formula (8) implies that the diameter $\mathcal{F}(x_n)$ converges to zero. As these closed sets are nested and since the space is complete, it follows that

$$\bigcap_{n \geq 0} \mathcal{F}(x_n) = \{\bar{x}\}. \tag{12}$$

Since $\bar{x} \in \mathcal{F}(x_0)$, the inequality (i) is satisfied. On the other hand, $\bar{x}$ belongs to all the $\mathcal{F}(x_n)$; it follow that $\mathcal{F}(\bar{x}) \subset \mathcal{F}(x_n)$ and consequently that

$$\mathcal{F}(\bar{x}) = \{\bar{x}\}. \tag{13}$$

It is clear tha if $x \neq \bar{x}$ then $x \notin \mathcal{F}(\bar{x})$, this means $f(x) + d(\bar{x}, x) > f(\bar{x})$. The part (ii) is satisfied. □

Theorem 2.4 above was applied by Caristi to determine a fixed point of a set-valued.

Theorem 2.5 (Caristi [10])**.** *Let $\mathcal{F}$ be a strict set-valued map of a complete metric space $\mathcal{X}$ into $\mathcal{P}_o(\mathcal{X})$. We suppose that there exist function $f : \mathcal{X} \longrightarrow [0, +\infty]$ is lower semi-continuous such that*

$$\forall x \in \mathcal{X}, \exists y \in \mathcal{F}(x), \quad f(y) + d(x, y) \leq f(x). \tag{14}$$

Then the set-valued map $\mathcal{F}$ has a fixed point.
If f is linked to $\mathcal{F}$ satisfy below

$$\forall x \in \mathcal{X}, \forall y \in \mathcal{F}(x), \quad f(y) + d(x, y) \leq f(x), \tag{15}$$

then $\mathcal{F}$ has a unique fixed point $\bar{x} \in \mathcal{X}$ and $\mathcal{F}(\bar{x}) = \{\bar{x}\}$.

Proof. Suppose that $\bar{x}$ satisfies part (ii) in Theorem 2.4, that is for $x \neq \bar{x}$, we have $f(\bar{x}) < f(x) + \epsilon d(x, \bar{x})$. With $\epsilon < 1$ and $\bar{y} \in \mathcal{F}(\bar{x})$ satisfies

$$f(\bar{y}) + \epsilon d(\bar{x}, \bar{y}) \leq f(\bar{y}) + d(\bar{x}, \bar{y}) \leq f(\bar{x}).$$

If $\bar{y} \neq \bar{x}$, inequality part (ii) in Theorem 2.4, with $x = \bar{y}$ implies that $f(\bar{y}) + d(\bar{x}, \bar{y}) \leq f(\bar{x}) < f(\bar{y}) + \epsilon d(\bar{y}, \bar{x})$ or $d(\bar{x}, \bar{y}) < \epsilon d(\bar{y}, \bar{x})$, which is impossible since $\epsilon < 1$. Thus should be, $\bar{y} = \bar{x}$. This means there is at least one such if condition (14) is satisfied, while all the $\bar{y} \in F(x)$ equal to $\bar{x}$ if condition (15) is satisfied. □

Discrete dynamical systems have widespread applications across various disciplines, particularly in the analysis and modeling of phenomena or processes occurring in discrete time steps. Economic growth models and market prediction analyses, for example, rely heavily on discrete dynamical systems. When generated by set-valued mappings, these systems model economic scenarios where market growth depends on multiple uncertain variables, such as government policies or consumer behavior.

3. Main Results

Let $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{P}_o(\mathcal{X})$ be set-valued maps. A discrete dynamical system is based on it if the relation system is defined

$$x_{n+1} \in \mathcal{F}(x_n), \tag{16}$$

where $n = 0, 1, \ldots$ and $x_0 = x$ is a fixed. Here, we assume that the realization of x_{n+1} can be any element of $\mathcal{F}(x_n)$. A sequence $\langle x_n \rangle$ of element $x_n \in X$ satisfying (16) is called "trajectory" of $\mathcal{F}$ starting at x.

In economics, fixed points are crucial, as equilibrium in many mathematical economic models is often described by a fixed point [11]. Arrow and Debreu were awarded the Nobel Prize for proving the existence of fixed points in such models. Debreu's model, in particular, is a special case of (16) [12, 13].

Next, we collect trajectories or orbits with initial state values as follows. A set

$$H(x) = \bigcup \{x_n \mid for \quad all \quad trajectories \quad starting \quad at \quad x\}$$

is called "orbit" starting at x.
Any trajectory $\langle x_n \rangle$ of F is trajectory of H in the sense that $x_{n+1} \in H(x_n)$; a trajectory of H may regarded as a sub-trajectory of F

Definition 3.1. *If $\langle x_n \rangle$ is trajectory of $\mathcal{F}$, the set*

$$L(x_n) = \bigcap_{n \geq 0} \left(\overline{\bigcup_{n \geq N} \langle x_n \rangle} \right)$$

is said to be the limit set of orbit.

By making use of the existence of a strict function associated with set-valued mappings, we obtained a theorem.

Theorem 3.2. *Suppose that $\mathcal{F} : \mathcal{X} \longrightarrow \wp(X)$ is the set-valued maps, and the set $\mathcal{X}$ is a complete metric space. Suppose there exist a function $g : \mathcal{X} \longrightarrow [0, +\infty]$ satisfying*

$$\forall x \in \mathcal{X}, \exists y \in \mathcal{F}(x), \quad g(y) + d(x, y) \leq g(x). \tag{17}$$

If the graph of $\mathcal{F}$ is closed, then for all $x_0 \in Dom(g)$, there exists a sequence $\langle x_n \rangle \subset \mathcal{X}$ satisfying (16) which converges to a fixed point of $\mathcal{F}$.

Proof. Suppose a point $x_0 \in Dom(f)$, by hypothesis there exists $x_1 \in \mathcal{F}(x_0)$ such that

$$d(x_1, x_0) \leq g(x_0) - g(x_1).$$

If the point $x_n \in \mathcal{X}$, then there exists $x_{n+1} \in \mathcal{F}(x_n)$ such that

$$d(x_{n+1}, x_n) \leq g(x_n) - g(x_{n+1}). \tag{18}$$

This implies that the sequence of positive number $\langle g(x_n) \rangle$ is decreasing; thus, it converges to a number, β. Adding the inequalities (18) from $n = p$ to $n = q - 1$, the triangle inequality implies that

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x_{p+1}) + d(x_{p+1}, x_{p+2}) + \cdots + d(x_{q-2}, x_{q-1}) + d(x_{q-1}, x_q) \\ &\leq (g(x_p) - g(x_{p+1})) + (g(x_{p+1}) - g(x_{p+2})) + \cdots \\ &\quad \cdots \quad + (g(x_{q-2}) - g(x_{q-1})) + (g(x_{q-1}) - g(x_q)) \\ &\leq g(x_p) - g(x_q). \end{aligned} \tag{19}$$

Since the term on the right tends to $(\beta - \beta) = 0$ as $p, q \longrightarrow \infty$, we deduce that the sequence x_n is the Cauchy sequence. Since the metric space $\mathcal{X}$ is complete, of course, the sequence x_n converges to $\bar{x} \in \mathcal{X}$.

Since the pairs $(x_n, x_{n+1}) \in \text{Graph}(\mathcal{F})$ converges to $(\bar{x}, \bar{x})$ and the Graph($\mathcal{F}$) is closed, the pairs $(\bar{x}, \bar{x})$ belongs to the graph of $\mathcal{F}$. The point $\bar{x} \in \mathcal{F}(\bar{x})$. In other words, $\bar{x}$ is a fixed point of $\mathcal{F}$. □

Theorem 3.2 is the result of modifications to Theorem 2.5 by adding sufficient conditions for closed set-valued mappings. These properties are used to investigate the existence of the system orbit generated by set-valued mappings.

The orbit resulting from the iteration of system (16) with the initial state x is a collection of orbits x_n, which may be finite or infinite in number. The following theorem shows that the limit point and the image of the intersection of orbits are elements and subsets of the collection of intersecting orbits.

Theorem 3.3. *If any sequence $\langle x_n \rangle$ of* (16) *we associate the set*

$$K(x_n) = \bigcap_{n \geq 0} \overline{H(x_n)}, \tag{20}$$

(i) Then the limit points of the sequence $\langle x_n \rangle$ belong to $K(x_n)$.

(ii) If $\mathcal{F}$ l.s.c, then $\mathcal{F}(K(x_n)) \subset K(x_n)$.

Proof. (i) We know that the set of limit points of $\langle x_n \rangle$ is the set

$$L(x_n) = \bigcap_{n \geq 0} \left(\overline{\bigcup_{n \geq N} \{x_n\}} \right).$$

Since $\bigcup_{n \geq N} \{x_n\} \subset H(x_N)$, it follow that

$$L(x_n) = \bigcap_{n \geq 0} \left(\overline{\bigcup_{n \geq N} \{x_n\}} \right) \subset \bigcap_{n \geq 0} H(x_n) \subset \bigcap_{n \geq 0} \overline{H(x_n)} = K(x_n).$$

It is proven that $L(x_n) \subset K(x_n)$.

(ii) Since image $\mathcal{F}(H(x)) \subset H(x)$, it follow that when $\mathcal{F}$ is l.s.c, $\mathcal{F}(\overline{H(x)}) \subset \overline{H(x)}$. Now we suppose that $y = \lim_n y_n$, where $y_n \in H(x)$ and $z \in \mathcal{F}(y)$. Since $\mathcal{F}$ is l.s.c, there exists $z_n \in \mathcal{F}(y_n) \subset H(x)$ such that z_n converges to z. Consequently, $z \in H(x)$. It then follows that $\mathcal{F}(L(x_n)) = \mathcal{F}(K(x_n)) \subset K(x_n)$.
It is proven that $\mathcal{F}(K(x_n)) \subset K(x_n)$.

□

We suppose that there exists a function $h : X \longrightarrow [0, +\infty]$ satisfying

$$\forall x \in \mathcal{X}, \quad \forall y \in \mathcal{F}(x), \quad h(y) + d(x, y) \leq h(x). \tag{21}$$

Theorem 3.4. *For each $x \in Dom(h)$, construct a sequence $\langle x_n \rangle$ trajectory of (16) with starting at $x_0 = x$ satisfying*

$$x_{n+1} \in H(x_n) \quad \textit{and} \quad h(x_{n+1}) \leq v(x_n) + 2^{-n} \tag{22}$$

where

$$v(x) = \inf\{h(y) \mid y \in H(x)\} \tag{23}$$

Then the sequence $\langle x_n \rangle$ converges to a limit $\bar{x}$.

Proof. For $x \in Dom(h)$ and assumption (21) implies that

$$\forall y \in H(x), \quad d(x, y) \leq h(x) - h(y).$$

Actually, $y = x_n$, where $x_n \in \mathcal{F}(x_{n-1}), \cdots, x_1 \in \mathcal{F}(x)$. Thus,

$$\begin{aligned} d(x, y) &\leq d(x, x_1) + d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, y) = \sum_{m=1}^{n} d(x_{m-1}, x_m) \\ &\leq \sum_{m=1}^{n} (h(x_{m-1}) - h(x_m)) \\ &= h(x_0) - h(x_n) \\ &= h(x) - h(y) \end{aligned}$$

From (23) of the function h, it follows that

$$\forall y \in H(x), \quad \inf_{y \in H(x)} d(x, y) \leq h(x) - \inf_{y \in H(x)} h(y) = h(x) - v(x),$$

hence that

$$Diam(H(x)) \leq 2(h(x) - v(x)).$$

We now choose the sequence $\langle x_n \rangle$ satisfying (22) and show that

$$v(x_{n+1}) \leq h(x_{n+1}) \leq v(x_n) + 2^{-n} \leq v(x_{n+1}) + 2^{-n}$$

or

$$0 \leq h(x_{n+1}) - v(x_{n+1}) \leq 2^{-n}.$$

Thus, the decreasing sequence of sets $\overline{H(x_n)}$, the diameter of which tends to zero, reduces to single point $\bar{x} \in H(x_n)$, whence $H(\bar{x}) \subset H(x_n)$, it follows that

$$H(\bar{x}) \subset \cap_{n \geq 0} \overline{H(x_n)} = \{\bar{x}\} \subset H(\bar{x})$$

or

$$H(\bar{x}) = \{\bar{x}\}$$

□

Theorem 3.4 shows that the iteration with the generating function of the collection of orbits, satisfying certain conditions (22) and (23), the orbits of the discrete dynamical system will converge to a unique fixed point for set-valued mappings.

4. Conclusion

In this section, we present the conclusion of the above results. By the intersection of the collected trajectory (orbit) solutions of the dynamical system and utilizing the Caristi mapping for set-valued mappings, it is obtained that a sequence of trajectories (orbits) converges to a fixed point (Theorem 3.2). The result of Theorem 3.3 leads to Theorem 3.4, which shows that the orbits of the system converge to a unique fixed point.

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