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On Sums Involving Polynomials and Generalized Fibonacci Sequences

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Abstract
This paper discovers a new identity of finite series generated by polynomial of real coefficients and certain generalized Fibonacci sequence. Let this generalized Fibonacci sequence be denoted by $(s_n)_{n\geq 0}$ and be defined with initial values $s_0 = c_0$, $s_1 = c_1$ and the recurrence relation $s_{n+1} = as_n + bs_{n-1}$ for all positive integers n, where a, b are positive integers, and c_0, c_1 are integers with $(c_0, c_1) \neq (0, 0)$. Then, we find an interesting identity of the series $\sum_{k=1}^{n} P(k)s_{k-1}$ where $P(x)$ is a polynomial of real coefficients. This series can be represented by identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(x)s_{n+1} + G_1(n)s_n + H_1(n)$ for all positive integers n, where $F_1(x)$, $G_1(x)$, and $H_1(x)$ are certain polynomials with real coefficients. Besides discovering this identity, there are some fabulous properties behind this identity which can be observed. We observe that the existence of triple $(F_1(x), G_1(x), H_1(x))$ satisfying this identity is a guarantee, but their uniqueness is not a guarantee. Therefore, some possible cases which derives the uniqueness and the non-uniqueness of triple $(F_1(x), G_1(x), H_1(x))$ are also studied.

Keywords: *Fibonacci sums, generalized Fibonacci sequences, polynomials of real coefficients, finite series*

1. Introduction

Generalized Fibonacci sequence is an interesting topic in recursive sequences, especially second-order linear recurrence sequences. It modifies the recurrence relation and two initial values of old Fibonacci sequence. We know that each term in old Fibonacci sequence is obtained by summing up two previous terms, so each term in generalized Fibonacci sequence is a linear combination of two previous terms. Also, instead of initial values 0 and 1, generalized Fibonacci sequence allows its two initial values with any of two numbers. For the literatures that discuss the generalized Fibonacci sequence, see [2], [3], [4], [5], [10], [12].

In this paper, we consider the generalized Fibonacci sequence $(s_n)_{n\geq 0}$ defined by the recursion $s_{n+1} = as_n + bs_{n-1}, \forall n \in \mathbb{N}$ and two initial values $s_0 = c_0, s_1 = c_1$ where $a, b \in \mathbb{N}; c_0, c_1 \in \mathbb{Z}$, $(c_0, c_1) \neq (0, 0)$. Many well-known recursive sequences are special cases of $(s_n)_{n\geq 0}$, such as Fibonacci sequence [6], Lucas sequence [6], Pell sequence [7], and Jacobsthal sequence [11]. If $(a, b, c_0, c_1) = (1, 1, 0, 1)$, then $(s_n)_{n\geq 0}$ becomes the Fibonacci sequence $(F_n)_{n\geq 0}$. If $(a, b, c_0, c_1) = (1, 1, 2, 1)$, then $(s_n)_{n\geq 0}$ becomes the Lucas sequence $(L_n)_{n\geq 0}$. If $(a, b, c_0, c_1) = (2, 1, 0, 1)$, then $(s_n)_{n\geq 0}$ becomes the Pell sequence $(P_n)_{n\geq 0}$. If $(a, b, c_0, c_1) = (1, 2, 0, 1)$, then $(s_n)_{n\geq 0}$ becomes the Jacobsthal sequence $(J_n)_{n\geq 0}$.

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Definition 1.1. *Let $(s_n)_{n\geq 0}$ be a generalized Fibonacci sequence defined by:*

$$s_0 = c_0, \;\; s_1 = c_1, \;\; \textit{and} \;\; (\forall n \in \mathbb{N})\; s_{n+1} = as_n + bs_{n-1}$$

with $a, b \in \mathbb{N}$; $c_0, c_1 \in \mathbb{Z}$; $(c_0, c_1) \neq (0,0)$.

The purpose of this paper is to state a new identity of the sum $\sum_{k=1}^{n} P(k)s_{k-1}$ where $P(x)$ is a real coefficient polynomial. Also, we study some interesting properties related to this identity. We first hypothesize that for a certain polynomial $P(x)$ in $\mathbb{R}[x]$, there always exist three polynomials $F_1(x), G_1(x), H_1(x) \in \mathbb{R}[x]$ (depending on $P(x)$) satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$ for all $n \in \mathbb{N}$.

As examples, when we replace $(s_n)_{n\geq 0}$ by Fibonacci sequence $(F_n)_{n\geq 0}$, the possible triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ for $P(x) = x$, $P(x) = x^2 + 2x$, $P(x) = 2x^2 + 1$ are respectively $(x-1, -1, 1)$, $(x^2+3, -2x+1, -3)$, $(2x^2-4x+11, -4x+6, -11)$. In other word, these represent the following identities in Example 1.2, holds for all $n \in \mathbb{N}$. One can show these three identities by using induction on n.

Example 1.2. *Identities for series* $\sum_{k=1}^{n} kF_{k-1}$, $\sum_{k=1}^{n}(k^2+2k)F_{k-1}$, $\sum_{k=1}^{n}(2k^2+1)F_{k-1}$:

$$\sum_{k=1}^{n} kF_{k-1} = (n-1)F_{n+1} - F_n + 1,$$

$$\sum_{k=1}^{n}(k^2+2k)F_{k-1} = (n^2+3)F_{n+1} + (-2n+1)F_n - 3,$$

$$\sum_{k=1}^{n}(2k^2+1)F_{k-1} = (2n^2-4n+11)F_{n+1} + (-4n+6)F_n - 11.$$

Some articles has mainly inspired us to find a new identity for the series $\sum_{k=1}^{d} P(k)s_{k-1}$, for example [8] and [1]. Respectively, they find the identities generated by series $\sum_{k=1}^{n} k^m F_k$ and $\sum_{k=1}^{n} k^m F_{k+r}$, where $(F_n)_{n\geq 0}$ are Fibonacci sequence. Ledin [8] proposes an identity for series $\sum_{k=1}^{n} k^m F_k$ with $m \in \mathbb{N}_0$. He presents that $\sum_{k=1}^{n} k^m F_k = P_2(m,n)F_{n+1} + P_1(m,n)F_n + C(m)$ where $P_1(m,n)$ and $P_2(m,n)$ are polynomials in n of degree m, and $C(m)$ is a real constant depending on m. Besides that, in [1], Brousseau discovers the general formula of series $\sum_{k=1}^{n} k^m F_{k+r}$ with $m, r \in \mathbb{N}$ by using a finite difference approach.

We are also motivated by problem 1410 Spring 2024 of Pi Mu Epsilon journal [9] which is proposed by Kenny B. Davenport. The problem considers Pell sequence $(P_n)_{n\geq 0}$ where $P_0 = 0$, $P_1 = 1$, $P_{k+2} = 2P_{k+1} + P_k$, $\forall k \geq 0$. There are two identities that must be proved:

$$2\sum_{k=1}^{n} kP_{k-1} = nP_{n+1} - (n+1)P_n; \quad 2\sum_{k=1}^{n} k^2 P_{k-1} = (n^2+1)P_{n+1} - (n^2+2n)P_n - 1. \tag{1}$$

These identities can be proved easily by induction on n, but the more challenging question in this problem is "can we conjecture what shape of the identity obtained by the series $\sum_{k=1}^{n} k^d P_{k-1}$?". Luckily, we find an interesting identity of the series $\sum_{k=1}^{n} k^d P_{k-1}$ for general $d \in \mathbb{N}_0$ and it generalizes two identities in (1). This identity is in the following theorem.

Theorem 1.3. *Let $d \in \mathbb{N}_0$, $(P_n)_{n\geq 0}$ be Pell sequence, and $A_d = (a_{i,j})_{i,j=1}^{d+1}$ be a $(d+1)\times(d+1)$ square matrix defined by:*

$$a_{i,j} = \begin{cases} 0 & \textit{if} \quad i > j \\ 2 & \textit{if} \quad i = j \\ (2^{j-i}+2)\binom{j-1}{i-1} & \textit{if} \quad i < j. \end{cases}$$

The matrix A_d is invertible because its determinant is 2^{d+1} which is non-zero. Therefore there exists an ordered $(d+1)$-tuple $(\lambda_0(d), \lambda_1(d), \ldots, \lambda_d(d)) \in \mathbb{R}^{d+1}$ which is a unique solution of *matrix equation*

$$A_d(\lambda_0(d), \lambda_1(d), \ldots, \lambda_d(d))^T = \left(\binom{d}{0}2^d, \binom{d}{1}2^{d-1}, \ldots, \binom{d}{d}2^0\right)^T.$$

Then the polynomials $F(x)$, $G(x)$, $H(x)$ defined by:

$$F(x) = \sum_{i=0}^{d} \lambda_i(d)x^i,\ G(x) = -(x+1)^d + \sum_{i=0}^{d}(x+1)^i\lambda_i(d),\ H(x) = 2^d - \sum_{i=0}^{d}(2^i+2)\lambda_i(d)$$

satisfy the identity

$$(\forall n \in \mathbb{N})\ \sum_{k=1}^{n} k^d P_{k-1} = F(n)P_{n+1} + G(n)P_n + H(n). \tag{2}$$

Example 1.4. *Some identities of $\sum_{k=1}^n k^d P_{k-1}$ for lower degrees d are:*

$$d = 0 \implies \sum_{k=1}^{n} P_{k-1} = \frac{1}{2}P_{n+1} - \frac{1}{2}P_n - \frac{1}{2},$$

$$d = 1 \implies \sum_{k=1}^{n} kP_{k-1} = \frac{1}{2}nP_{n+1} - \frac{1}{2}(n+1)P_n,$$

$$d = 2 \implies \sum_{k=1}^{n} k^2P_{k-1} = \frac{1}{2}(n^2+1)P_{n+1} - \frac{1}{2}(n^2+2n)P_n - \frac{1}{2},$$

$$d = 3 \implies \sum_{k=1}^{n} k^3P_{k-1} = \frac{1}{2}(n^3+3n-3)P_{n+1} - \frac{1}{2}(n^3+3n^2+1)P_n + \frac{3}{2},$$

$$d = 4 \implies \sum_{k=1}^{n} k^4P_{k-1} = \frac{1}{2}(n^4+6n^2-12n+13)P_{n+1} - \frac{1}{2}(n^4+4n^3+4n-6)P_n - \frac{13}{2}.$$

Identity (2) can be proved by using induction. It can be generalized to be Theorem 2.5 preceded by Lemma 2.4. As our attempt to generalize $(P_n)_{n\geq 0}$ to $(s_n)_{n\geq 0}$, we can generalize the identity (2) to be identity (5). Identity (5) states that for non-negative integer d, the following identity holds: $\sum_{k=1}^n k^d s_{k-1} = F_{0,d}(n)s_{n+1} + G_{0,d}(n)s_n + H_{0,d}(n)$ for all positive integers n, where $F_{0,d}(x)$, $G_{0,d}(x)$, and $H_{0,d}(x)$ are certain polynomials with real coefficients. Later, an example of such polynomials $F_{0,d}(x)$, $G_{0,d}(x)$, $H_{0,d}(x)$ is stated in Theorem 2.5. One example of triple $(F_{0,d}(x), G_{1,d}(x), H_{1,d}(x)) \in \mathbb{R}[x]^3$ which satisfies the identity $\sum_{k=1}^n k^d s_{k-1} = F_{0,d}(n)s_{n+1} + G_{0,d}(n)s_n + H_{0,d}(n)$, $\forall n \in \mathbb{N}$ is $(F_{1,d}(x), G_{1,d}(x), H_{1,d}(x))$ where $F_{1,d}(x)$, $G_{1,d}(x)$, and $H_{1,d}(x)$ are polynomials of real coefficients defined in Theorem 2.5. It has shown the existence of triple $(F_{0,d}(x), G_{0,d}(x), H_{0,d}(x)) \in \mathbb{R}[x]^3$ which satisfies the identity $\sum_{k=1}^n k^d s_{k-1} = F_{0,d}(n)s_{n+1} + G_{0,d}(n)s_n + H_{0,d}(n)$ for all $n \in \mathbb{N}$. Later, we can extend $\sum_{k=1}^n k^d s_{k-1}$ into the more general series $\sum_{k=1}^n P(k)s_{k-1}$ where $P(x)$ is arbitrary polynomial in $\mathbb{R}[x]$. In other word, for every polynomials $P(x) \in \mathbb{R}[x]$, there exist polynomials with real coefficients $F_1(x)$, $G_1(x)$, $H_1(x)$ satisfying the identity $\sum_{k=1}^n P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$ for all positive integers n.

The triple $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ which satisfies the identity $\sum_{k=1}^n P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ is ensured to be exist, but not always unique. The non-uniqueness of $(F_1(x), G_1(x), H_1(x))$ can happen, for example, when we set $(a, b, c_0, c_1) = (1, 12, 1, 4)$ to $(s_n)_{n\geq 0}$, i.e., $(s_n)_{n\geq 0}$ with recurrence relation $s_{n+1} = s_n + 12s_{n-1}, \forall n \in \mathbb{N}$ and initial values $s_0 = 1$, $s_1 = 4$. Then we also set $P(x) = 72x + 18$, so the following three identities hold for all positive integers n and one can show these by induction.

$$\sum_{k=1}^{n}(72k+18)s_{k-1} = s_{n+1} + (24n-6)s_n + 2,$$
$$\sum_{k=1}^{n}(72k+18)s_{k-1} = (6n+1)s_{n+1} - 6s_n + 2,$$
$$\sum_{k=1}^{n}(72k+18)s_{k-1} = 3n^2 s_{n+1} - (12n^2-24n+2)s_n + 2.$$

From these facts, if $(s_n)_{n\geq 0}$ has the constraint $(a,b,c_0,c_1) = (1,12,1,4)$, then the possible triples $(F_1(x),G_1(x),H_1(x)) \in \mathbb{R}[x]^3$ which satisfy the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n), \forall n \in \mathbb{N}$ are (1, $24x-6$, 2), ($6x+1$, -6, 2), and ($3x^2$, $-12x^2+24x-2$, 2). These things have shown us that for a polynomial $P(x) \in \mathbb{R}[x]$, the triple $(F_1(x),G_1(x),H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ can not be guaranteed to be unique. Therefore, we also investigate what types of sequence $(s_n)_{n\geq 0}$ that implies the uniqueness and the non-uniqueness of triple $(F_1(x),G_1(x),H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$ for all $n \in \mathbb{N}$.

We intend to consider two types of $(s_n)_{n\geq 0}$ which imply the uniqueness and the non-uniqueness of this triple respectively. These two types of sequence $(s_n)_{n\geq 0}$ are: (i) $(s_n)_{n\geq 0}$ with both of $2c_1 - (a+\sqrt{a^2+4b})c_0$ and $2c_1 - (a-\sqrt{a^2+4b})c_0$ are non-zero, and (ii) $(s_n)_{n\geq 0}$ with one of $2c_1 - (a+\sqrt{a^2+4b})c_0$ or $2c_1 - (a-\sqrt{a^2+4b})c_0$ equals zero. When both of $2c_1 - (a+\sqrt{a^2+4b})c_0$ and $2c_1 - (a-\sqrt{a^2+4b})c_0$ are non-zero, by the help of Theorem 2.9, we ensure that the such triple $(F_1(x),G_1(x),H_1(x))$ is unique with dependent of $P(x)$. If we set $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ (where $m \in \mathbb{N}_0$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}$), we obtain $F_1(x) = \sum_{d=0}^{m} a_d F_{1,d}(x)$, $G_1(x) = \sum_{d=0}^{m} a_d G_{1,d}(x)$, $H_1(x) = \sum_{d=0}^{m} a_d H_{1,d}(x)$ where $F_{1,d}(x)$, $G_{1,d}(x)$, $H_{1,d}(x)$ are polynomials which stated in Theorem 2.5. On the other hand, when one of $2c_1 - (a+\sqrt{a^2+4b})c_0$ or $2c_1 - (a-\sqrt{a^2+4b})c_0$ equals zero, so for a single polynomial $P(x) \in \mathbb{R}[x]$, there are infinitely many triples $(F_1(x),G_1(x),H_1(x))$ in $\mathbb{R}[x]^3$ such that the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ holds.

2. Some Important Facts

For the convenient of writings, let us denote j_1 and j_2 as follows.

Definition 2.1. $j_1 = a + \sqrt{a^2+4b}$ *and* $j_2 = a - \sqrt{a^2+4b}$.

We consider again the sequence $(s_n)_{n\geq 0}$, as defined in Definition (1.1), with the recurrence relation $s_{n+1} = as_n + bs_{n-1}$, $\forall n \in \mathbb{N}$ and initial values $s_0 = c_0$, $s_1 = c_1$ where $(a,b) \in \mathbb{N}^2$ and $(c_0,c_1) \in \mathbb{Z}^2 \backslash \{(0,0)\}$. Then the sequence $(s_n)_{n\geq 0}$ is explicitly given by the Binet-type formula

$$s_n = \frac{1}{2\sqrt{a^2+4b}}\left((2c_1 - j_2c_0)\left(\frac{j_1}{2}\right)^n - (2c_1 - j_1c_0)\left(\frac{j_2}{2}\right)^n\right), \quad \forall n \in \mathbb{N}_0 \tag{3}$$

where $j_1/2 = \left(a+\sqrt{a^2+4b}\right)/2$ and $j_2/2 = \left(a-\sqrt{a^2+4b}\right)/2$ are roots of quadratic equation $x^2 - ax - b = 0$. Both of a and b are positive integers, so the discriminant of $x^2 - ax - b = 0$ is $a^2 + 4b$. It is clear that $a^2 + 4b > a^2 > 0$. Therefore, all roots of $x^2 - ax - b = 0$ are real, distinct, and non-zero.

Another property about $(s_n)_{n\geq 0}$ is that there exists $N \in \mathbb{N}$ so that $s_n \neq 0$ for all $n \geq N$. This property is useful to verify that s_{n+1}/s_n will converge to a number as n goes to ∞. If there does not exist $m \in \mathbb{N}$ so that $s_m = 0$, then there exists $N = 1$ so that $s_n \neq 0$ for all

$n \geq N = 1$. If there exists $m \in \mathbb{N}$ so that $s_m = 0$, by formula (3), we have

$$\begin{aligned}(2c_1 - j_2c_0)\left(\frac{j_1}{2}\right)^m = (2c_1 - j_1c_0)\left(\frac{j_2}{2}\right)^m &\iff (2c_1 - j_2c_0)j_1^m = (2c_1 - j_1c_0)j_2^m \\ &\iff (j_1^m - j_2^m)c_1 = \frac{1}{2}(j_1^{m-1} - j_2^{m-1})j_1j_2c_0 \\ &\iff (j_1^m - j_2^m)c_1 = -2(j_1^{m-1} - j_2^{m-1})bc_0.\end{aligned}$$

If $c_0 = 0$ then $(j_1^m - j_2^m)c_1 = 0$. Since $j_1 \neq j_2$, then $c_1 = 0$. It implies $(c_0, c_1) = (0, 0)$, a contradiction. Therefore c_0 must be non-zero and we get

$$\frac{-c_1}{2bc_0} = \frac{j_1^{m-1} - j_2^{m-1}}{j_1^m - j_2^m}. \tag{4}$$

Let $\xi : \mathbb{N} \to \mathbb{R}$ be a function defined by $\xi(n) = \frac{j_1^{n-1} - j_2^{n-1}}{j_1^n - j_2^n}$, $\forall n \in \mathbb{N}$. We can observe that ξ is strictly increasing and thus the equation (4) must have exactly one solution in m. Then it implies $s_n \neq 0$ for all $n \geq m+1$ (in this case, we can take $N = m+1$).

Lemma 2.2. *There exists $N \in \mathbb{N}$ so that $s_n \neq 0$ for all $n \geq N$.*

Lemma 2.3. *$2c_1 - j_1c_0$ and $2c_1 - j_2c_0$ can not be simultaneously equal to zero.*

Proof. Assume the contrary that $2c_1 - j_1c_0 = 2c_1 - j_2c_0 = 0$. By equation (3), $s_n = 0$ for all $n \in \mathbb{N}_0$, contradicting the Lemma 2.2. Hence, $(2c_1 - j_1c_0, 2c_1 - j_2c_0) \neq (0, 0)$. ☐

If $2c_1 - j_1c_0 = 0$, then $2c_1 - j_2c_0 \neq 0$ and $s_n = \frac{2c_1 - j_2c_0}{2\sqrt{a^2+4b}}\left(\frac{j_1}{2}\right)^n$ for all $n \in \mathbb{N}_0$.

If $2c_1 - j_2c_0 = 0$, then $2c_1 - j_1c_0 \neq 0$ and $s_n = \frac{-2c_1 + j_1c_0}{2\sqrt{a^2+4b}}\left(\frac{j_2}{2}\right)^n$ for all $n \in \mathbb{N}_0$.

If $2c_1 - j_1c_0$ and $2c_1 - j_2c_0$ are non-zero: Let us consider the subsequence $(s_n)_{n \geq N}$ with $s_n \neq 0$, $\forall n \geq N$ for some $N \in \mathbb{N}$. By setting n going to ∞ for s_{n+1}/s_n, we get

$$\begin{aligned}\lim_{n\to\infty} \frac{s_{n+1}}{s_n} &= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(2c_1 - j_2c_0)j_1^{n+1} - (2c_1 - j_1c_0)j_2^{n+1}}{(2c_1 - j_2c_0)j_1^n - (2c_1 - j_1c_0)j_2^n} \\ &= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(2c_1 - j_2c_0)j_1 - (2c_1 - j_1c_0)j_2\left(\frac{j_2}{j_1}\right)^n}{(2c_1 - j_2c_0) - (2c_1 - j_1c_0)\left(\frac{j_2}{j_1}\right)^n} \\ &= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(2c_1 - j_2c_0)j_1}{2c_1 - j_2c_0} \qquad \left(\text{because } \left|\frac{j_2}{j_1}\right| < 1\right) \\ &= \frac{j_1}{2}.\end{aligned}$$

Hence the sequence $(\frac{s_{n+1}}{s_n})_{n \geq N}$ converges to $j_1/2 = \left(a + \sqrt{a^2+4b}\right)/2$ for this case.

Lemma 2.4. *Let d be a non-negative integer. Then there exist the polynomials $F_{0,d}(x)$, $G_{0,d}(x)$, $H_{0,d}(x)$ with real coefficients satisfying*

$$(\forall n \in \mathbb{N}) \quad \sum_{k=1}^{n} k^d s_{k-1} = F_{0,d}(n)s_{n+1} + G_{0,d}(n)s_n + H_{0,d}(n) \tag{5}$$

An example of ordered triple of polynomials $(F_{0,d}(x), G_{0,d}(x), H_{0,d}(x))$ that satisfies the identity (5) is stated in Theorem 2.5 as follows.

Theorem 2.5. *Suppose that* $d \in \mathbb{N}_0$. *Let* $B_d = (b_{i,j})_{i,j=1}^{d+1}$ *be a* $(d+1) \times (d+1)$ *real square matrix with*

$$b_{i,j} = \begin{cases} 0 & \text{if } \ i > j \\ a+b-1 & \text{if } \ i = j \\ (2^{j-i}b + a)\binom{j-1}{i-1} & \text{if } \ i < j. \end{cases}$$

Then, we get the following statements.
(i). There exists uniquely an ordered (d+1)-tuple $(b_0(d), b_1(d), \ldots, b_d(d)) \in \mathbb{R}^{d+1}$ *satisfying the matrix equation*

$$B_d\,(b_0(d), b_1(d), \ldots, b_d(d))^T = \left(\binom{d}{0}2^d, \binom{d}{1}2^{d-1}, \ldots, \binom{d}{d}2^0\right)^T \tag{6}$$

(ii). Let $(b_0(d), b_1(d), \ldots, b_d(d))$ *be the unique solution of* (6).
Then the triple of polynomials $(F_{1,d}(x), G_{1,d}(x), H_{1,d}(x))$ *defined by*

$$F_{1,d}(x) = \sum_{i=0}^{d} b_i(d)x^i, \quad G_{1,d}(x) = -(x+1)^d + \left(\sum_{i=0}^{d}(x+1)^i b_i(d)\right)b,$$

$$\text{and} \quad H_{1,d}(x) = c_0 + 2^d c_1 - (ac_1 + bc_0)\sum_{i=0}^{d} b_i(d) - bc_1 \sum_{i=0}^{d} 2^i b_i(d)$$

is an example of triple $(F_{0,d}(x), G_{0,d}(x), H_{0,d}(x))$ *which satisfy the identity* (5).

Proof. (i). The determinant of B_d is $(a+b-1)^{d+1} \neq 0$, then the matrix B_d is invertible and it implies that the equation (6) has exactly one solution $(b_0(d), b_1(d), \ldots, b_d(d))$ in $\mathbb{C}^{d+1}$. Furthermore,

$$(b_0(d), b_1(d), \ldots, b_d(d))^T = B_d^{-1}\left(\binom{d}{0}2^d, \binom{d}{1}2^{d-1}, \ldots, \binom{d}{d}2^0\right)^T.$$

Since $B_d^{-1} \in \mathbb{M}_{d+1}(\mathbb{R})$ and $\left(\binom{d}{0}2^d, \binom{d}{1}2^{d-1}, \ldots, \binom{d}{d}2^0\right)^T \in \mathbb{M}_{(d+1)\times 1}(\mathbb{R})$, then $(b_0(d), b_1(d), \ldots, b_d(d)) \in \mathbb{R}^{d+1}$ and the result follows.

(ii). We show this part by using induction. When $n = 1$, the LHS of (5) is $\sum_{k=1}^{1} k^d s_{k-1} = s_0 = c_0$ and the RHS of (5) is $F_{1,d}(1)s_2 + G_{1,d}(1)s_1 + H_{1,d}(1) = \left(\sum_{i=0}^{d} b_i(d)\right)(ac_1 + bc_0) + \left(-2^d + \left(\sum_{i=0}^{d} 2^i b_i(d)\right)b\right)c_1 + c_0 + 2^d c_1 - (ac_1 + bc_0)\sum_{i=0}^{d} b_i(d) - bc_1\sum_{i=0}^{d} 2^i b_i(d) = c_0$. In this case, the LHS and RHS of (5) are being same, so (5) satisfies for $n = 1$.

Assume that the identity (5) satisfies for $n = m$ for some $m \in \mathbb{N}$, so

$$\sum_{k=1}^{m} k^d s_{k-1} = F_{1,d}(m)s_{m+1} + G_{1,d}(m)s_m + H_{1,d}(m) \tag{7}$$

We have to show that the identity (5) also holds for $n = m+1$. Before we show it, we would like to show that for all $x \in \mathbb{R}$,

$$F_{1,d}(x+1) : (F_{1,d}(x) - G_{1,d}(x+1)) : \left((x+1)^d + G_{1,d}(x)\right) = 1 : a : b \tag{8}$$

,i.e., $F_{1,d}(x) - G_{1,d}(x+1) = aF_{1,d}(x+1)$ and $(x+1)^d + G_{1,d}(x) = bF_{1,d}(x+1)$.
By definitions of $F_{1,d}(x)$ and $G_{1,d}(x)$ in Theorem 2.5 part (ii), we have

$$(x+1)^d + G_{1,d}(x) = \left(\sum_{i=0}^{d}(x+1)^i b_i(d)\right)b = bF_{1,d}(x+1) \tag{9}$$

Observe that the matrix equation (6) is equivalent to

$$(\forall k = 0, 1, 2, \ldots, d) \quad \sum_{i=k}^{d}(b \cdot 2^{i-k} + a)\binom{i}{k}b_i(d) \;=\; \binom{d}{k}2^{d-k} + b_k(d) \tag{10}$$

We also observe that the degrees of $F_{1,d}(x) - G_{1,d}(x+1)$ and $aF_{1,d}(x+1)$ are not more than d. For all $k = 0, 1, 2, \ldots, d$; the coefficients of x^k in polynomials $F_{1,d}(x) - G_{1,d}(x+1)$ and $aF_{1,d}(x+1)$ are respectively

$$b_k(d) + \binom{d}{k} 2^{d-k} - \left(\sum_{i=k}^{d} 2^{i-k} \binom{i}{k} b_i(d) \right) b \qquad \text{and} \qquad \left(\sum_{i=k}^{d} \binom{i}{k} b_i(d) \right) a \tag{11}$$

Because of identity (10), two expressions in (11) are equivalent. It implies that

$$F_{1,d}(x) - G_{1,d}(x+1) = aF_{1,d}(x+1) \tag{12}$$

By (9) and (12), we have shown the condition (8) as desired.

We remember the recurrence relation of $(s_n)_{n \geq 0}$: $s_{n+1} = as_n + bs_{n-1}$ for all $n \in \mathbb{N}$. The coefficients of s_{n+1}, s_n, and s_{n-1} in this recurrence relation have the ratio $1 : a : b$ which is same as the ratio in (8). Hence, the following identity holds for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

$$F_{1,d}(x+1)s_{n+1} = \left(F_{1,d}(x) - G_{1,d}(x+1)\right) s_n + \left((x+1)^d + G_{1,d}(x)\right) s_{n-1} \tag{13}$$

Setting $x := m$, $n := m+1$ to (13) yields

$$F_{1,d}(m+1)s_{m+2} = \left(F_{1,d}(m) - G_{1,d}(m+1)\right) s_{m+1} + \left((m+1)^d + G_{1,d}(m)\right) s_m$$

and equivalently,

$$(m+1)^d s_m = F_{1,d}(m+1)s_{m+2} + \left(G_{1,d}(m+1) - F_{1,d}(m)\right) s_{m+1} - G_{1,d}(m)s_m \tag{14}$$

Summing up the equations (7) and (14) yields

$$\sum_{k=1}^{m+1} k^d s_{k-1} = F_{1,d}(m+1)s_{m+2} + G_{1,d}(m+1)s_{m+1} + H_{1,d}(m+1)$$

and hence the identity (5) holds for $n = m+1$.

In conclusion, the identity (5) holds for all $n \in \mathbb{N}$. □

Example 2.6. *Some identities of* $\sum_{k=1}^{n} k^d s_{k-1}$ *for lower degrees* d *are:*

$$\sum_{k=1}^{n} s_{k-1} = \frac{s_{n+1}}{a+b-1} + \frac{(1-a)s_n}{a+b-1} + c_0 + c_1 - \frac{ac_1 + bc_0 + bc_1}{a+b-1},$$

$$\sum_{k=1}^{n} ks_{k-1} = \left(\frac{x}{a+b-1} + \frac{a-2}{(a+b-1)^2} \right) s_{n+1} + \left(\frac{(1-a)x}{a+b-1} + \frac{-a^2+2a-b-1}{(a+b-1)^2} \right) s_n$$
$$+ c_0 + 2c_1 - \frac{(ac_1+bc_0)(2a+b-3) + bc_1(3a+2b-4)}{(a+b-1)^2},$$

$$\sum_{k=1}^{n} k^2 s_{k-1} = \left(\frac{x^2}{a+b-1} + \frac{(2a-4)x}{(a+b-1)^2} + \frac{a^2-3a+4b-ab+4}{(a+b-1)^3} \right) s_{n+1}$$
$$+ \left(\frac{(1-a)x^2}{a+b-1} + \frac{(-2a^2+4a-2b-2)x}{(a+b-1)^2} + \frac{4a^2b+b^3+3ab^2-11ab-2b^2+9b}{(a+b-1)^3} - 1 \right) s_n$$
$$- \frac{(ac_1+bc_0)(4a^2+b^2+3ab-11a-2b+9) \; + \; bc_1(9a^2+4b^2+11ab-23a-12b+16)}{(a+b-1)^3}$$
$$+ c_0 + 4c_1.$$

We note that, because of Theorem 2.5, it should imply that for a polynomial $P(x) \in \mathbb{R}[x]$, there exist three polynomials $F_1(x), G_1(x), H_1(x) \in \mathbb{R}[x]$ so that $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$ for all positive integers n. Then, the new problem for us is "for a polynomial $P(x) \in \mathbb{R}[x]$, can we ensure that the triple of polynomials $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ which satisfies the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n), \forall n \in \mathbb{N}$ is unique ?". Therefore, for the next steps, we will investigate what cases that imply the uniqueness of such triple $(F_1(x), G_1(x), H_1(x))$. Although some cases do not imply the uniqueness of $(F_1(x), G_1(x), H_1(x))$, but indeed in those cases, there are infinitely many triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ which satisfy the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1}+G_1(n)s_n+H_1(n), \forall n \in \mathbb{N}$.

Lemma 2.7. *If $P(x)$ is a polynomial with real coefficients and $\lim\limits_{\substack{n\to\infty,\\ n\in\mathbb{N}}} P(n) = 0$, then $P(x)$ is zero polynomial.*

Proof. If $P(x)$ is not constant, then $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ for some $m \in \mathbb{N}_0$ and for some $a_0, a_1, \ldots, a_m \in \mathbb{R}$ with $a_m \neq 0$. Observe that for all $x \in \mathbb{R}^+$, we have

$$P(x) = a_m x^m \cdot \left(1 + \sum_{k=1}^{m} \frac{a_{m-k}}{x^k a_m}\right).$$

The condition $\lim\limits_{n\to\infty,\, n\in\mathbb{N}} P(n) = 0$ implies that

$$0 = \lim_{\substack{n\to\infty,\\ n\in\mathbb{N}}} |P(n)| = \lim_{\substack{n\to\infty,\\ n\in\mathbb{N}}} |a_m n^m| \cdot \lim_{\substack{n\to\infty,\\ n\in\mathbb{N}}} \left|1 + \sum_{k=1}^{m} \frac{a_{m-k}}{n^k a_m}\right| = \lim_{\substack{n\to\infty,\\ n\in\mathbb{N}}} |a_m n^m|.$$

It is impossible because $a_m \neq 0$ should imply $\lim\limits_{\substack{n\to\infty,\\ n\in\mathbb{N}}} |a_m n^m| = \infty$.

If $P(x)$ is a constant polynomial, let $P(x) = c$, $\forall x \in \mathbb{R}$. Then $0 = \lim\limits_{\substack{n\to\infty,\\ n\in\mathbb{N}}} P(n) = \lim\limits_{\substack{n\to\infty,\\ n\in\mathbb{N}}} c = c$, so $P(x)$ is zero polynomial.

In conclusion, $P(x)$ must be zero polynomial. □

Lemma 2.8. *Let k be a real number with $k \in \mathbb{R} \setminus [-1, 1]$. For all polynomials $P(x)$ and $Q(x)$ with real coefficients where $Q(x)$ is not zero polynomial, we have the limit*

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{P(n)/Q(n)}{k^n} = 0.$$

Proof. Consider an arbitrary $m \in \mathbb{N}_0$. For all positive integers $n > \frac{(m+2)!}{(\ln|k|)^{m+2}}$, we have

$$0 < \frac{n^{m+1}}{|k|^n} = \frac{n^{m+1}}{\sum_{a=0}^{\infty} \frac{(n\ln|k|)^a}{a!}} < \frac{n^{m+1}}{\frac{(n\ln|k|)^{m+2}}{(m+2)!}} = \frac{(m+2)!}{n(\ln|k|)^{m+2}} < 1 \implies 0 < \frac{n^m}{|k|^n} < \frac{1}{n}.$$

Since $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} 0 = \lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{1}{n} = 0$, by Squeeze theorem we obtain $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{n^m}{|k|^n} = 0$, then $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{n^m}{k^n} = 0$.
Here we get the fact that for all $m \in \mathbb{N}_0$, the following limit holds:

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{n^m}{k^n} = 0.$$

Let us consider arbitrarily the polynomials $P(x) \in \mathbb{R}[x]$ and $Q(x) \in \mathbb{R}[x] \setminus \{0\}$, so there exists $M \in \mathbb{N}$ such that $Q(x) \neq 0$ for all real $x \geq M$. If $P(x)$ is the zero polynomial, then

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{P(n)/Q(n)}{k^n} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} 0 = 0.$$

If $P(x)$ is not zero polynomial, then

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{P(n)/Q(n)}{n^{|\deg(P)-\deg(Q)|+1}} = 0 \quad \text{and} \quad \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{n^{|\deg(P)-\deg(Q)|+1}}{k^n} = 0$$

where deg(...) denotes the degree of a polynomial. Hence,

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{P(n)/Q(n)}{k^n} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{P(n)/Q(n)}{n^{|\deg(P)-\deg(Q)|+1}} \cdot \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{n^{|\deg(P)-\deg(Q)|+1}}{k^n} = 0\cdot 0 = 0.$$

This completes the proof. □

Theorem 2.9. *Let $\gamma_1(x)$, $\gamma_2(x)$, $\gamma_3(x)$ be polynomials with real coefficients. Let $2c_1 - j_1c_0$ and $2c_1 - j_2c_0$ be non-zero. If $\gamma_1(n)s_{n+1} + \gamma_2(n)s_n + \gamma_3(n) = 0$ holds for all $n \in \mathbb{N}$, then $\gamma_1(x)$, $\gamma_2(x)$, $\gamma_3(x)$ are zero polynomials.*

Proof. We know from Lemma 2.2 that there exists a number $N \in \mathbb{N}$ so that $s_n \neq 0$ for all $n \geq N$. Dividing both sides of identity $\gamma_1(n)s_{n+1} + \gamma_2(n)s_n + \gamma_3(n) = 0$ by s_n over integers $n \geq N$ implies

$$(\forall n \in \mathbb{N},\ n \geq N)\quad \gamma_1(n)\frac{s_{n+1}}{s_n} + \gamma_2(n) + \frac{\gamma_3(n)}{s_n} = 0 \tag{15}$$

Later, we also note the identity

$$(\forall n \in \mathbb{N},\ n \geq N)\quad \gamma_1(n)\left(\frac{-s_{n+1}}{s_n} + \frac{j_1}{2}\right) = \frac{-\gamma_1(n)(2c_1 - j_1c_0)\sqrt{a^2+4b}}{(2c_1 - j_2c_0)\,(j_1/j_2)^n - (2c_1 - j_1c_0)} \tag{16}$$

Observe that $j_1/j_2 < -1 < j_2/j_1 < 0$ and $j_1/2 > 1$. Then, by Lemma 2.8, we get

$$\begin{aligned}
\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\gamma_3(n)}{s_n} &= \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{2\sqrt{a^2+4b}\cdot\gamma_3(n)}{\left((2c_1 - j_2c_0) - (2c_1 - j_1c_0)\,(j_2/j_1)^n\right)(j_1/2)^n}\\
&= \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{2\sqrt{a^2+4b}}{\left((2c_1 - j_2c_0) - (2c_1 - j_1c_0)\,(j_2/j_1)^n\right)} \cdot \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\gamma_3(n)}{(j_1/2)^n}\\
&= \frac{2\sqrt{a^2+4b}}{2c_1 - j_2c_0}\cdot 0 = 0.
\end{aligned}$$

Summing up the limit $n \to \infty$ of (15) and (16) and then applying Lemma 2.8 yields

$$\begin{aligned}
\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left(\gamma_1(n)\frac{j_1}{2} + \gamma_2(n)\right) &= \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{-\gamma_1(n)(2c_1 - j_1c_0)\sqrt{a^2+4b}}{(2c_1 - j_2c_0)\,(j_1/j_2)^n - (2c_1 - j_1c_0)}\\
&= \frac{-(2c_1 - j_1c_0)\sqrt{a^2+4b}}{2c_1 - j_2c_0} \cdot \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\gamma_1(n)}{(j_1/j_2)^n} = 0.
\end{aligned}$$

Since $\gamma_1(x)\frac{j_1}{2} + \gamma_2(x) \in \mathbb{R}[x]$, by Lemma 2.7, then $\gamma_1(x)\frac{j_1}{2} + \gamma_2(x)$ is a zero polynomial. Substituting $\gamma_2(x) = -\frac{j_1}{2}\gamma_1(x), \forall x \in \mathbb{R}$ to identity $\gamma_1(n)s_{n+1} + \gamma_2(n)s_n + \gamma_3(n) = 0, \forall n \in \mathbb{N}$ yields

$$\begin{aligned}
(\forall n \in \mathbb{N})\quad 0 = \gamma_1(n)s_{n+1} + \gamma_2(n)s_n + \gamma_3(n) &= \gamma_1(n)\cdot\left(s_{n+1} - \frac{j_1}{2}s_n\right) + \gamma_3(n)\\
&= \frac{2c_1 - j_1c_0}{2}\left(\frac{j_2}{2}\right)^n \gamma_1(n) + \gamma_3(n)
\end{aligned}$$

$$\Longrightarrow \qquad (\forall n \in \mathbb{N})\quad \frac{2c_1 - j_1c_0}{2}\left(\frac{j_2}{2}\right)^n \gamma_1(n) + \gamma_3(n) = 0 \tag{17}$$

If one of $\gamma_1(x)$ or $\gamma_3(x)$ is a zero polynomial, by equation (17) it implies that both of $\gamma_1(x)$ and $\gamma_3(x)$ are zero polynomials. Then γ_2 is a zero polynomial too, and the theorem is done.
If both of $\gamma_1(x)$ and $\gamma_3(x)$ are not zero polynomials, then there exists $M_1 \in \mathbb{N}$ such that $\gamma_1(x) \neq 0$ and $\gamma_3(x) \neq 0$ for all $x \geq M_1$. From the identity (17), we have

$$(\forall n \in \mathbb{N}, n \geq M_1)\qquad \frac{\gamma_3(n)}{\gamma_1(n)} = \frac{-(2c_1 - j_1c_0)}{2}\left(\frac{j_2}{2}\right)^n \tag{18}$$

then

$$(\forall n \in \mathbb{N}, n \geq M_1) \quad \left|\frac{\gamma_3(n)}{\gamma_1(n)}\right| = \frac{1}{2}|2c_1 - j_1c_0|\left|\frac{j_2}{2}\right|^n \tag{19}$$

Let us observe (18) and (19) into 3 possibilities: $b - a > 1$, $b - a = 1$, and $b - a < 1$.

Case $b - a > 1$:
Observe that $j_2/2 < -1$. Therefore, by (18) and Lemma 2.8,

$$\frac{-(2c_1 - j_1c_0)}{2} = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\gamma_3(n)/\gamma_1(n)}{(j_2/2)^n} = 0$$

then $2c_1 - j_1c_0 = 0$, a contradiction.

Case $b - a = 1$:
We now have $j_2/2 = -1$, so the identities (18) and (19) respectively become

$$(\forall n \in \mathbb{N}, n \geq M_1) \quad \frac{\gamma_3(n)}{\gamma_1(n)} = \frac{-(2c_1 - j_1c_0)}{2}(-1)^n \tag{20}$$

and

$$(\forall n \in \mathbb{N}, n \geq M_1) \quad \left|\frac{\gamma_3(n)}{\gamma_1(n)}\right| = \frac{1}{2}|2c_1 - j_1c_0| \tag{21}$$

From (21), it indicates that either $\gamma_3(n) = \frac{1}{2}(2c_1 - j_1c_0)\gamma_1(n)$ for infinitely many integers n or $\gamma_3(n) = -\frac{1}{2}(2c_1 - j_1c_0)\gamma_1(n)$ for infinitely many integers n.
If $\gamma_3(n) = \frac{1}{2}(2c_1 - j_1c_0)\gamma_1(n)$ for infinitely many integers n, it implies $\gamma_3(x) = \frac{1}{2}(2c_1 - j_1c_0)\gamma_1(x)$ for all $x \in \mathbb{R}$. Therefore, we can rewrite (20) by

$$(\forall n \in \mathbb{N}, n \geq M_1) \quad \frac{2c_1 - j_1c_0}{2} = \frac{-(2c_1 - j_1c_0)}{2}(-1)^n \tag{22}$$

Setting to (22) when n is even yields that $2c_1 - j_1c_0 = 0$, a contradiction.
If $\gamma_3(n) = -\frac{1}{2}(2c_1 - j_1c_0)\gamma_1(n)$ for infinitely many integers n, then $\gamma_3(x) = -\frac{1}{2}(2c_1 - j_1c_0)\gamma_1(x)$ for all $x \in \mathbb{R}$. So we can rewrite (20) by

$$(\forall n \in \mathbb{N}, n \geq M_1) \quad -\frac{1}{2}(2c_1 - j_1c_0) = \frac{-(2c_1 - j_1c_0)}{2}(-1)^n \tag{23}$$

Setting to (23) when n is odd yields that $2c_1 - j_1c_0 = 0$, a contradiction.

Case $b - a < 1$:
We have that $2/j_2 < -1$. By identity (18) and Lemma 2.8,

$$\frac{2}{2c_1 - j_1c_0} = -\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\gamma_1(n)/\gamma_3(n)}{(2/j_2)^n} = 0$$

which is impossible since $\frac{2}{2c_1 - j_1c_0}$ itself is non-zero.

In conclusion, $\gamma_1(x)$, $\gamma_2(x)$, $\gamma_3(x)$ must be zero polynomials. □

Theorem 2.10. *Let $\gamma_1(x)$, $\gamma_2(x)$, $\gamma_3(x)$ be polynomials with real coefficients, $0 \in \{2c_1 - j_1c_0, 2c_1 - j_2c_0\}$, and $\gamma_1(n)s_{n+1} + \gamma_2(n)s_n + \gamma_3(n) = 0$ for all $n \in \mathbb{N}$.*
(i). If $2c_1 - j_1c_0 = 0$, then $\gamma_2(x) \equiv \frac{-j_1}{2} \cdot \gamma_1(x)$ and $\gamma_3(x) \equiv 0$.
(ii). If $2c_1 - j_2c_0 = 0$, then $\gamma_2(x) \equiv \frac{-j_2}{2} \cdot \gamma_1(x)$ and $\gamma_3(x) \equiv 0$.

Proof. (i). If $2c_1 - j_1c_0 = 0$, then $s_n = (j_1/2)^n K_1$ for all $n \in \mathbb{N}_0$, where $K_1 = \frac{2c_1 - j_2c_0}{2\sqrt{a^2+4b}}$ is a non-zero real constant. The identity $\gamma_1(n)s_{n+1} + \gamma_2(n)s_n + \gamma_3(n) = 0$, $\forall n \in \mathbb{N}$ becomes

$$\left(\gamma_1(n)\frac{j_1}{2} + \gamma_2(n)\right)\left(\frac{j_1}{2}\right)^n K_1 + \gamma_3(n) = 0, \quad \forall n \in \mathbb{N} \tag{24}$$

$$\iff \quad \gamma_1(n)\frac{j_1}{2} + \gamma_2(n) = \frac{-1}{K_1} \cdot \frac{\gamma_3(n)}{(j_1/2)^n}, \quad \forall n \in \mathbb{N}.$$

Observe that $j_1/2 > 1$. By Lemma 2.8, we have

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left(\gamma_1(n)\frac{j_1}{2} + \gamma_2(n)\right) = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{-1}{K_1} \cdot \frac{\gamma_3(n)}{(j_1/2)^n} = 0.$$

Since $\gamma_1(x)\frac{j_1}{2} + \gamma_2(x) \in \mathbb{R}[x]$, by Lemma 2.7, it leads to the result that $\frac{j_1}{2} \cdot \gamma_1(x) + \gamma_2(x)$ is zero polynomial, then $\gamma_2(x) \equiv \frac{-j_1}{2} \cdot \gamma_1(x)$. Also, equation (24) implies $\gamma_3(x) \equiv 0$.

(ii). If $2c_1 - j_2 c_0 = 0$, we have $s_n = (j_2/2)^n K_2$ for all $n \in \mathbb{N}$, where $K_2 = \frac{-2c_1 + j_1 c_0}{2\sqrt{a^2+4b}}$ is a non-zero real constant. Therefore, we can rewrite the identity $\gamma_1(n)s_{n+1} + \gamma_2(n)s_n + \gamma_3(n) = 0$, $\forall n \in \mathbb{N}$ by

$$\left(\gamma_1(n)\frac{j_2}{2} + \gamma_2(n)\right)\left(\frac{j_2}{2}\right)^n K_2 + \gamma_3(n) = 0, \quad \forall n \in \mathbb{N} \tag{25}$$

Let us denote the polynomial $\gamma_1(x)\frac{j_2}{2} + \gamma_2(x)$ by $\gamma_4(x)$. The identity (25) is equivalent to

$$\gamma_4(n)\left(\frac{j_2}{2}\right)^n K_2 + \gamma_3(n) = 0, \quad \forall n \in \mathbb{N} \tag{26}$$

Note that if either $\gamma_3(x)$ or $\gamma_4(x)$ is zero polynomial, then both of them becomes zero polynomials. Hence $\gamma_2(x) \equiv \frac{-j_2}{2} \cdot \gamma_1(x)$ and $\gamma_3(x) \equiv 0$, done.

If $\gamma_3(x)$ and $\gamma_4(x)$ are not zero polynomials, then there exists $M_2 \in \mathbb{N}$ in such a way that $\gamma_3(x) \neq 0$, $\gamma_4(x) \neq 0$ for all real $x \geq M_2$. So the identity (26) implies the following two identities:

$$(\forall n \in \mathbb{N}, n \geq M_2) \quad \frac{\gamma_3(n)}{\gamma_4(n)} = -K_2\left(\frac{j_2}{2}\right)^n \tag{27}$$

$$(\forall n \in \mathbb{N}, n \geq M_2) \quad \left|\frac{\gamma_3(n)}{\gamma_4(n)}\right| = |K_2|\left|\frac{j_2}{2}\right|^n \tag{28}$$

We intend to investigate (27) and (28) in 3 cases: $b > a+1$, $b = a+1$, $b < a+1$.
Case $b > a+1$: We have $j_2/2 < -1$. Hence by (27) and Lemma 2.8, it leads to the result

$$-K_2 = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\gamma_3(n)/\gamma_4(n)}{(j_2/2)^n} = 0,$$

it is a contradiction.

Case $b = a+1$: In this case, $j_2/2 = -1$. The identities (27) and (28) can be rewritten as the following two:

$$(\forall n \in \mathbb{N}, n \geq M_2) \quad \frac{\gamma_3(n)}{\gamma_4(n)} = -K_2\,(-1)^n \tag{29}$$

$$(\forall n \in \mathbb{N}, n \geq M_2) \quad \left|\frac{\gamma_3(n)}{\gamma_4(n)}\right| = |K_2| \tag{30}$$

The identity (30) gives us the fact that either $\gamma_3(n) = K_2\gamma_4(n)$ for infinitely many integers n or $\gamma_3(n) = -K_2\gamma_4(n)$ for infinitely many integers n.

If $\gamma_3(n) = K_2\gamma_4(n)$ for infinitely many integers n, it implies that $\gamma_3(x) = K_2\gamma_4(x)$ for all real x. Substituting it to (29) and setting when n is even yields that $K_2 = 0$, a contradiction.

If $\gamma_3(n) = -K_2\gamma_4(n)$ for infinitely many integers n, then $\gamma_3(x) = -K_2\gamma_4(x)$ for all real x. Substituting it to (29) and setting when n is odd yields that $K_2 = 0$, a contradiction.

Case $b < a+1$: In this case, we have the fact $2/j_2 < -1$. By (27) and Lemma 2.8, we obtain

$$\frac{-1}{K_2} = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\gamma_4(n)/\gamma_3(n)}{(2/j_2)^n} = 0,$$

but it is impossible.
In conclusion, the result must be $\gamma_2(x) \equiv \frac{-j_2}{2} \cdot \gamma_1(x)$ and $\gamma_3(x) \equiv 0$. □

3. Three Main Theorems

This section presents three main theorems of this paper, explaining the identity and properties of series $\sum_{k=1}^{n} P(x)s_{k-1}$ for a polynomial $P(x)$ in $\mathbb{R}[x]$. These theorems give the main result about finite series involving a polynomial of real coefficients and certain generalized Fibonacci sequence.

Theorem 3.1 is a development of Lemma 2.4 and Theorem 2.5, because Lemma 2.4 and Theorem 2.5 only represent the summation $\sum_{k=1}^{n} k^d s_{k-1}$ for arbitrary $d \in \mathbb{N}_0$, while Theorem 3.1 represents the summation $\sum_{k=1}^{n} P(k)s_{k-1}$ for all general polynomials $P(x) \in \mathbb{R}[x]$. For an arbitrary $P(x) \in \mathbb{R}[x]$, we can find the existence of polynomials $F_1(x), G_1(x), H_1(x) \in \mathbb{R}[x]$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$.

Theorem 3.1. *Let $P(x)$ be a polynomial with real coefficients. Then there exists a triple of polynomials $(F_1(x), G_1(x), H_1(x))$ with real coefficients so that $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$.*

Proof. Consider a polynomial $P(x) \in \mathbb{R}[x]$.
We can state that $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ for some $m \in \mathbb{N}_0$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}$. An example of triple $(F_1(x), G_1(x), H_1(x))$ which satisfies the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ is

$$F_1(x) = \sum_{d=0}^{m} a_d F_{1,d}(x); \quad G_1(x) = \sum_{d=0}^{m} a_d G_{1,d}(x); \quad H_1(x) = \sum_{d=0}^{m} a_d H_{1,d}(x)$$

where $F_{1,d}(x)$, $G_{1,d}(x)$, $H_{1,d}(x)$ are polynomials as defined in Theorem 2.5 part (ii).
Hence, the result follows. □

Not only that, for a polynomial $P(x) \in \mathbb{R}[x]$, we also determine how many triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$. Here, we get the result that if both of $2c_1 - j_1c_0$ and $2c_1 - j_2c_0$ are non-zero, then the triple $(F_1(x), G_1(x), H_1(x))$ is unique. If one of $2c_1 - j_1c_0$ or $2c_1 - j_2c_0$ equals zero, then there are infinitely many triples $(F_1(x), G_1(x), H_1(x))$.

Theorem 3.2. *Let $P(x)$ be a polynomial of real coefficients. If both of $2c_1 - j_1c_0$ and $2c_1 - j_2c_0$ are non-zero, then the triple $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ is unique.*

Proof. Suppose that $(F_2(x), G_2(x), H_2(x))$ and $(F_3(x), G_3(x), H_3(x))$ are the ordered solution of $(F_1(x), G_1(x), H_1(x))$ which satisfy the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$. Then we have

$$\sum_{k=1}^{n} P(k)s_{k-1} = F_2(n)s_{n+1} + G_2(n)s_n + H_2(n) = F_3(n)s_{n+1} + G_3(n)s_n + H_3(n), \quad \forall n \in \mathbb{N}$$

then

$$(F_2 - F_3)(n)s_{n+1} + (G_2 - G_3)(n)s_n + (H_2 - H_3)(n) = 0, \quad \forall n \in \mathbb{N}.$$

Since $(F_2-F_3)(x)$, $(G_2-G_3)(x)$, $(H_2-H_3)(x)$ are polynomials with real coefficients, by Theorem 2.9 we get that $(F_2-F_3)(x)$, $(G_2-G_3)(x)$, $(H_2-H_3)(x)$ are identically zero. Therefore $F_2(x)$, $G_2(x)$, $H_2(x)$ are identically equal to $F_3(x)$, $G_3(x)$, $H_3(x)$ respectively.
Hence the triple of polynomials $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ which satisfies the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ is unique. □

Theorem 3.3. *Let $P(x)$ be a polynomial of real coefficients. If either $2c_1 - j_1c_0$ or $2c_1 - j_2c_0$ is equal to 0, then there are infinitely many triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$.*

Proof. Define that Ω_1 and Ω_2 are the sets of all triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ with the constraints $2c_1 - j_1c_0 = 0$ and $2c_1 - j_2c_0 = 0$ respectively. Let $P(x) = a_mx^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$ for some $m \in \mathbb{N}_0$ and $a_0, a_1, \cdots, a_m \in \mathbb{R}$.

If $2c_1 - j_1c_0 = 0$. For all $(F_2(x), G_2(x), H_2(x))$ and $(F_3(x), G_3(x), H_3(x))$ in Ω_1, we have

$$\sum_{k=1}^{n} P(k)s_{k-1} = F_2(n)s_{n+1} + G_2(n)s_n + H_2(n) = F_3(n)s_{n+1} + G_3(n)s_n + H_3(n), \quad \forall n \in \mathbb{N}$$

then

$$(F_2-F_3)(n)s_{n+1} + (G_2-G_3)(n)s_n + (H_2-H_3)(n) = 0, \quad \forall n \in \mathbb{N}.$$

then, by Theorem 2.10,

$$\frac{-j_1}{2} \cdot (F_2-F_3)(x) \equiv (G_2-G_3)(x) \quad \text{and} \quad (H_2-H_3)(x) \equiv 0$$

$$\iff \quad \frac{j_1}{2} \cdot (F_2+G_2)(x) \equiv \frac{j_1}{2} \cdot F_3(x) + G_3(x) \quad \text{and} \quad H_2(x) \equiv H_3(x).$$

It is clear that $(\sum_{d=0}^{m} a_dF_{1,d}(x),\ \sum_{d=0}^{m} a_dG_{1,d}(x),\ \sum_{d=0}^{m} a_dH_{1,d}(x))$ is an element in Ω_1, so it leads to the fact that for all $(F(x), G(x), H(x))$ in Ω_1,

$$\frac{j_1}{2} \cdot F(x) + G(x) \equiv \frac{j_1}{2} \sum_{d=0}^{m} a_dF_{1,d}(x) + \sum_{d=0}^{m} a_dG_{1,d}(x) \quad \text{and} \quad H(x) \equiv \sum_{d=0}^{m} a_dH_{1,d}(x).$$

Therefore

$$\Omega_1 = \{(F(x), \frac{-j_1}{2} \cdot F(x) + \frac{j_1}{2} \sum_{d=0}^{m} a_dF_{1,d}(x) + \sum_{d=0}^{m} a_dG_{1,d}(x), \sum_{d=0}^{m} a_dH_{1,d}(x)) \bigm| F(x) \in \mathbb{R}[x]\}$$

and it is obvious that $|\Omega_1| = \infty$.

It is also similar when $2c_1 - j_2c_0 = 0$. We will directly get

$$\Omega_2 = \{(F(x), \frac{-j_2}{2} \cdot F(x) + \frac{j_2}{2} \sum_{d=0}^{m} a_dF_{1,d}(x) + \sum_{d=0}^{m} a_dG_{1,d}(x), \sum_{d=0}^{m} a_dH_{1,d}(x)) \bigm| F(x) \in \mathbb{R}[x]\}.$$

and then $|\Omega_2| = \infty$.
Hence, $|\Omega_1| = \infty$ and $|\Omega_2| = \infty$ complete the proof. □

4. Summary

The following are summary of the results in this paper.

First. Let $P(x)$ be a polynomial of real coefficients and $(s_n)_{n\geq 0}$ be generalized Fibonacci sequence with $s_0 = c_0$, $s_1 = c_1$, and $s_{n+1} = as_n + bs_{n-1}$ for all $n \in \mathbb{N}$ where $(a, b) \in \mathbb{N}^2$ and $(c_0, c_1) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$. Then there exist polynomials $F_1(x)$, $G_1(x)$, $H_1(x)$ of real coefficients satisfying the identity

$$\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n), \quad \forall n \in \mathbb{N}.$$

Since $P(x) \in \mathbb{R}[x]$, let us suppose that $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ for some $m \in \mathbb{N}_0$ and some $a_0, a_1, \ldots, a_m \in \mathbb{R}$. Therefore, an example of triple $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ is

$$F_1(x) = \sum_{d=0}^{m} a_d F_{1,d}(x); \quad G_1(x) = \sum_{d=0}^{m} a_d G_{1,d}(x); \quad H_1(x) = \sum_{d=0}^{m} a_d H_{1,d}(x)$$

where the polynomials $F_{1,d}(x)$, $G_{1,d}(x)$, $H_{1,d}(x)$ for each $d \in \mathbb{N}_0$ are defined by

$$F_{1,d}(x) = \sum_{i=0}^{d} b_i(d)x^i, \quad G_{1,d}(x) = -(x+1)^d + \left(\sum_{i=0}^{d}(x+1)^i b_i(d)\right) b,$$

$$\text{and} \quad H_{1,d}(x) = c_0 + 2^d c_1 - (ac_1 + bc_0)\sum_{i=0}^{d} b_i(d) - bc_1 \sum_{i=0}^{d} 2^i b_i(d)$$

where $(b_0(d), b_1(d), \ldots, b_d(d))$ is the unique solution of equation

$$B_d\,(b_0(d), b_1(d), \ldots, b_d(d))^T = \left(\binom{d}{0}2^d, \binom{d}{1}2^{d-1}, \ldots, \binom{d}{d}2^0\right)^T$$

where $B_d = (b_{i,j})_{i,j=1}^{d+1}$ is a $(d+1) \times (d+1)$ real square matrix with

$$b_{i,j} = \begin{cases} 0 & \text{if} \quad i > j \\ a + b - 1 & \text{if} \quad i = j \\ (2^{j-i}b + a)\binom{j-1}{i-1} & \text{if} \quad i < j. \end{cases}$$

Second. Let $P(x)$ be a polynomial of real coefficients and $(s_n)_{n\geq 0}$ be generalized Fibonacci sequence with $s_0 = c_0$, $s_1 = c_1$, and $s_{n+1} = as_n + bs_{n-1}$ for all $n \in \mathbb{N}$ where $(a, b) \in \mathbb{N}^2$ and $(c_0, c_1) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$. The triple $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity

$$\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n), \quad \forall n \in \mathbb{N}.$$

in ensured to be exist, but is not guaranteed to be unique. We consider two possible cases of $(s_n)_{n\geq 0}$ which imply the uniqueness and the non-uniqueness of such triple $(F_1(x), G_1(x), H_1(x))$, as follows.
CASE 1: For $(s_n)_{n\geq 0}$ with constraints $2c_1 - (a + \sqrt{a^2+4b})c_0 \neq 0$ and $2c_1 - (a - \sqrt{a^2+4b})c_0 \neq 0$, then the triple $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ which satisfies the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ is unique. Furthermore, if we set $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ for some $m \in \mathbb{N}_0$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}$, the unique triple $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]$ which satisfy the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ is

$$F_1(x) = \sum_{d=0}^{m} a_d F_{1,d}(x); \quad G_1(x) = \sum_{d=0}^{m} a_d G_{1,d}(x); \quad H_1(x) = \sum_{d=0}^{m} a_d H_{1,d}(x)$$

where the polynomials $F_{1,d}(x)$, $G_{1,d}(x)$, $H_{1,d}(x)$ for each $d \in \mathbb{N}_0$ are defined by

$$F_{1,d}(x) = \sum_{i=0}^{d} b_i(d)x^i, \quad G_{1,d}(x) = -(x+1)^d + \left(\sum_{i=0}^{d}(x+1)^i b_i(d)\right) b,$$

$$\text{and} \quad H_{1,d}(x) = c_0 + 2^d c_1 - (ac_1 + bc_0)\sum_{i=0}^{d} b_i(d) - bc_1 \sum_{i=0}^{d} 2^i b_i(d)$$

where $(b_0(d), b_1(d), \ldots, b_d(d))$ is the unique solution of equation

$$B_d\,(b_0(d), b_1(d), \ldots, b_d(d))^T = \left(\binom{d}{0}2^d, \binom{d}{1}2^{d-1}, \ldots, \binom{d}{d}2^0\right)^T$$

where $B_d = (b_{i,j})_{i,j=1}^{d+1}$ is a $(d+1)\times(d+1)$ real square matrix with

$$b_{i,j} = \begin{cases} 0 & \text{if} \quad i > j \\ a+b-1 & \text{if} \quad i = j \\ (2^{j-i}b + a)\binom{j-1}{i-1} & \text{if} \quad i < j. \end{cases}$$

CASE 2: For $(s_n)_{n\geq 0}$ with either $2c_1 - (a + \sqrt{a^2+4b})c_0 = 0$ or $2c_1 - (a - \sqrt{a^2+4b})c_0 = 0$, then there are infinitely many triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$. Suppose that Ω_1 and Ω_2 are the sets of all triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ when $2c_1 - (a + \sqrt{a^2+4b})c_0 = 0$ and $2c_1 - (a - \sqrt{a^2+4b})c_0 = 0$ respectively. If we set $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ for some $m \in \mathbb{N}_0$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}$, then we have

$$\Omega_1 = \{(F(x), \frac{-j_1}{2}\cdot F(x) + \frac{j_1}{2}\sum_{d=0}^{m} a_d F_{1,d}(x) + \sum_{d=0}^{m} a_d G_{1,d}(x), \sum_{d=0}^{m} a_d H_{1,d}(x)) \mid F(x) \in \mathbb{R}[x]\}$$

and

$$\Omega_2 = \{(F(x), \frac{-j_2}{2}\cdot F(x) + \frac{j_2}{2}\sum_{d=0}^{m} a_d F_{1,d}(x) + \sum_{d=0}^{m} a_d G_{1,d}(x), \sum_{d=0}^{m} a_d H_{1,d}(x)) \mid F(x) \in \mathbb{R}[x]\}$$

where $j_1 = a + \sqrt{a^2+4b}$, $j_2 = a - \sqrt{a^2+4b}$, and the polynomials $F_{1,d}(x)$, $G_{1,d}(x)$, $H_{1,d}(x)$ for each $d \in \mathbb{N}_0$ are defined by

$$F_{1,d}(x) = \sum_{i=0}^{d} b_i(d)x^i, \quad G_{1,d}(x) = -(x+1)^d + \left(\sum_{i=0}^{d}(x+1)^i b_i(d)\right) b,$$

$$\text{and} \quad H_{1,d}(x) = c_0 + 2^d c_1 - (ac_1 + bc_0)\sum_{i=0}^{d} b_i(d) - bc_1 \sum_{i=0}^{d} 2^i b_i(d)$$

where $(b_0(d), b_1(d), \ldots, b_d(d))$ is the unique solution of equation

$$B_d\,(b_0(d), b_1(d), \ldots, b_d(d))^T = \left(\binom{d}{0}2^d, \binom{d}{1}2^{d-1}, \ldots, \binom{d}{d}2^0\right)^T$$

where $B_d = (b_{i,j})_{i,j=1}^{d+1}$ is a $(d+1)\times(d+1)$ real square matrix with

$$b_{i,j} = \begin{cases} 0 & \text{if} \quad i > j \\ a+b-1 & \text{if} \quad i = j \\ (2^{j-i}b + a)\binom{j-1}{i-1} & \text{if} \quad i < j. \end{cases}$$

5. Conclusion

This research yields the new identity of series $\sum_{k=1}^{n} P(k)s_{k-1}$ where $P(x)$ is a polynomial in $\mathbb{R}[x]$ and $(s_n)_{n\geq 0}$ is generalized Fibonacci sequence with $s_0 = c_0$, $s_1 = c_1$, and $s_{n+1} = as_n + bs_{n-1}, \forall n \in \mathbb{N}$ where $(a, b) \in \mathbb{N}^2$ and $(c_0, c_1) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$. For a polynomial $P(x) \in \mathbb{R}[x]$, we have the identity $\sum_{k=1}^{n} P(x)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ for some polynomials $F_1(x)$, $G_1(x)$, $H_1(x)$ in $\mathbb{R}[x]$. Besides that, we also determine how many triple $(F_1(x), G_1(x), H_1(x) \in \mathbb{R}[x]^3$ satisfying the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ for a polynomial $P(x) \in \mathbb{R}[x]$. If neither $2c_1 - (a + \sqrt{a^2 + 4b})c_0$ nor $2c_1 - (a - \sqrt{a^2 + 4b})c_0$ equals zero, then for a polynomial $P(x) \in \mathbb{R}[x]$, the triple $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ which satisfy the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$ is unique. If either $2c_1 - (a + \sqrt{a^2 + 4b})c_0$ or $2c_1 - (a - \sqrt{a^2 + 4b})c_0$ equals zero, then for a polynomial $P(x) \in \mathbb{R}[x]$, there are infinitely many triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]^3$ which satisfy the identity $\sum_{k=1}^{n} P(k)s_{k-1} = F_1(n)s_{n+1} + G_1(n)s_n + H_1(n)$, $\forall n \in \mathbb{N}$.

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