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Soft Union Bi-Quasi-Interior Ideals of Semigroups

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Abstract
The concept of the soft union (S-uni) bi-quasi-interior (BQI) ideal of semigroups is proposed in this study, along with its equivalent definition. We derive the relationships between S-uni ideals and S-uni BQI ideal. The S-uni BQI ideal is shown to be S-uni bi-ideal, left ideal, right ideal, interior ideal, quasi-ideal, bi-interior ideal, left/right bi-quasi ideal, and left/right quasi-interior ideal. It is shown that certain additional requirements, such as regularity or right/left simplicity, are necessary for the converses, and counterexamples are given to demonstrate that the converses are not true. Additionally, it is demonstrated that the soft anti characteristic function of a subsemigroup of a semigroup is an S-uni BQI ideal if the subsemigroup itself is a BQI ideal, and vice versa. Consequently, a significant connection between soft set theory and classical semigroup theory is established. Additionally, it is demonstrated that while the finite soft OR-products and union of S-uni BQI ideals are also S-uni BQI ideals, the intersection and finite soft AND-products are not. A broad conceptual characterization and analysis of S-uni BQI ideals are presented in this paper. This paper presents a generalization of many S-uni ideals in the literature making important contributions from this perspective.

Keywords: *Soft set, Semigroup, Bi-quasi-interior ideal, Soft uni bi-quasi-interior ideal, Regular semigroup.*

1. Introduction

The abstract algebraic foundation for ”memoryless” systems that reset with each iteration is provided by semigroups, which are essential in many branches of mathematics. In applied mathematics, semigroups-which were first examined in the early 1900s-are crucial instruments for examining linear time-invariant processes. Furthermore, as finite semigroups and finite automata are closely related, studying them is essential to theoretical computer science.

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Besides, many mathematicians have concentrated their studies on generalizing ideals in algebraic structures. Actually, generalizing ideals is necessary for more research related to algebraic structures. Numerous mathematicians have made important contributions by using the idea and characteristics of these extended ideals, providing fresh perspectives and descriptions of algebraic structures. Dedekind first proposed the notion of ideals in algebraic number theory, and Noether expanded it to include associative rings. Both one-sided and two-sided ideals remain essential to the study of ring theory, and the idea of a one-sided ideal in any algebraic structure may be viewed as an extension of the original ideal concept.

Bi-ideals for semigroups were first proposed by Good and Hughes [1] in 1952. The notion of quasi-ideals for semigroups was initially defined by Steinfeld [2], who then expanded it to include rings. Bi-ideals are a further generalization of quasi-ideals, which are a generalization of right and left ideals. Lajos [3] first introduced the concept of interior ideals, and Szasz [4, 5] later developed it further. The ideal notion was generalized to create the idea of interior ideals. Bi-interior, bi-quasi, bi-quasi-interior, weak-interior, tri, tri-quasi ideals are some of the new semigroup ideal types that Rao [6–11] proposed and which build upon preexisting ones. Additionally, Baupradist et al. [12] introduced the idea of essential ideals in semigroups. The concept of "almost" ideals was put forth as a more comprehensive extension of different kinds of ideals, and their properties as well as their connections to other related ideals were carefully examined. In this regard, the concept of almost ideals in semigroups was first introduced in [13]. A subsequent work [14] expanded the notion of bi-ideals to almost bi-ideals in semigroups. The idea of almost quasi-ideals was initially presented in [15], and in [16], the study of almost interior ideals and weakly almost interior ideals of semigroups led to the further development and exploration of both almost ideals and interior ideals in semigroups. In [17–19], the authors introduced the concepts like almost subsemigroups/bi-quasi-ideal interior ideals/ bi-interior ideals/ BQI ideals of semigroups, respectively. Furthermore, a variety of fuzzy almost ideal types for semigroups were investigated in [15, 17–22].

Molodtsov [23] presented the "Soft Set Theory" (SS Theory) in 1999 to deal with uncertainty-related problems and provide suitable solutions. Since then, many significant research have concentrated on different SS notions, especially on the operations carried out on them. Maji et al. [24] specified certain operations for SS and put out a number of definitions. Pei and Mia [25] and Ali et al. [26] proposed many operations on SS operations. SS operations were also studied by Sezgin and Atagün [27] in detail. We refer to [28–39] for further information on SS operations, which have become more and more common since their beginnings. Çağman and Enginoğlu made additional modifications to the concepts related to SS [40]. Later, Çağman et al. [41] developed the idea of soft int groups, which led to the exploration of various soft algebraic systems. By applying SS to semigroup theory, Sezgin [42] proposed soft union (S-uni) semigroups, left ($\mathcal{L}$) ideals, right ($\mathcal{R}$) ideals, two-sided ideals, Sezer et al [43] proposed interior ideals, quasi-ideal ideals, and (generalized) by providing an in-depth analysis of their fundamental properties. In the context of S-uni substructures of semigroups, Sezgin et al. [44] and Sezgin and Orbay [45] classified several types of semigroups, including semisimple semigroups, duo semigroups, $\mathcal{R}$ ($\mathcal{L}$) zero semigroups, $\mathcal{R}$ ($\mathcal{L}$) simple semigroups, semi-lattices of $\mathcal{L}$ ($\mathcal{R}$) simple semigroups, semi-lattices of $\mathcal{L}$ ($\mathcal{R}$) groups, and semi-lattices of groups. Soft intersection almost ideals, as a generalization of various types of soft intersection ideals, were introduced and examined in [46–57]. The soft versions of different algebraic structures were explored in [58–70].

As a generalization of the bi-ideal, quasi-ideal, interior ideal, bi-quasi ideal, and bi-interior ideal of a semigroup, Rao [7] presented the notion of the BQI ideal and investigated the characteristics of these ideals as well as their connections. As a generalization of bi-ideal, quasi-ideal, interior ideal, bi-quasi ideal, and bi-interior ideal in Γ-semirings, the idea of BQI ideal is examined by Rao [71]. These principles' characteristics are examined, as well as how they relate to one another. The characteristics of the regularSoft Union Bi-Quasi-Interior ($\mathcal{R}$) and simple Γ-semirings are explored, and the requirements for a Γ-semiring to be either simple or $\mathcal{R}$ are investigated. The characteristics of BQI ideals in Γ-semigroups and semirings, as well as their connections with other ideals, were also examined by Rao [72, 73]. The regularity and simplicity characteristics of semirings and Γ-semigroups are described, and the requirements for being $\mathcal{R}$ or simple are found.

This paper defines the term ”Soft union bi-quasi ideal” (S-uni BQI Ideal) of a semigroup for SS theory and thoroughly examines its characteristics and connections to other S-uni ideals. It is concluded that if a subsemigroup of a semigroup is a BQI ideal, then its anti soft characteristic function is also an S-uni BQI ideal, and the converse is also true. SS theory and classical semigroup theory are significantly connected by this important theorem. Moreover, every S-uni bi-ideal, ideal, interior ideal, quasi-ideal, bi-interior ideal, bi-quasi ideal, and quasi-interior ideal of a semigroup is an S-uni BQI ideal, as can be seen by examining the relationships of S-uni BQI ideal of a semigroup with other S-uni ideals. Counterexamples are given to demonstrate that the opposite is not true. Additionally, the conditions necessary for the converses to hold are obtained. Furthermore, the connections between S-uni BQI ideals and soft set operations are examined, along with ideas such as soft anti image and soft inverse image. Four sections make up the framework of the paper. While Section 1 provides a broad overview of the subject, Section 2 delves into the basic ideas of semigroups and SS ideals, as well as the definitions and consequences that go along with them. Using specific examples, we present the idea of S-uni BQI ideals in Section 3 and look at their characteristics and relationships to other kinds of S-uni ideals. A summary of our findings and some future study areas are discussed in Section 4.

2. Methods

Throughout this paper, S denotes a semigroup. A nonempty subset $\mathbb{K}$ of S is called a subsemigroup of S if $\mathbb{K}\mathbb{K} \subseteq \mathbb{K}$, is called a bi-ideal of S if $S\mathbb{K} \subseteq \mathbb{K}$ and $\mathbb{K}S\mathbb{K} \subseteq \mathbb{K}$, is called an interior ideal of S if $S\mathbb{K}S \subseteq \mathbb{K}$, and is called a quasi-ideal of S if $\mathbb{K}S \cup S\mathbb{K} \subseteq \mathbb{K}$. A subsemigroup $\mathbb{K}$ of S is called a BQI ideal of S if $\mathbb{K}S\mathbb{K} \subseteq \mathbb{K}$ [7]. A semigroup S is called a regular ($\mathcal{R}$) semigroup, if for all $x \in S$, there exists an element $y \in S$ such that $x = xyx$.

Theorem 2.1. *[74] Let S be a semigroup. Then,*

(1) S is $\mathcal{L}$ ($\mathcal{R}$) simple if and only if (iff) $Sa = S$ ($aS = S$) for all $a \in S$. That is, for every $a, b \in S$, there exists $c \in S$ such that $b = ca$ ($b = ac$)
(2) S is simple iff S is a group. (both $\mathcal{L}$ and $\mathcal{R}$ simple)

Definition 2.2. *[23,40] Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U, and $\mathbb{K} \subseteq E$. The soft set (SS) $f_{\mathbb{K}}$ over U is a function such that $f_{\mathbb{K}} \colon E \to P(U)$, where for all $y \notin \mathbb{K}$, $f_{\mathbb{K}}(y) = \emptyset$. That is,*

$$f_{\mathbb{K}} = \{(y, f_{\mathbb{K}}(y)) : y \in E, f_{\mathbb{K}}(y) \in P(U)\}$$

The set of all SS over U is designated by $S_E(U)$ throughout this paper.

Definition 2.3. *[40] Let $f_{\mathbb{K}} \in S_E(U)$. If $f_{\mathbb{K}}(y) = \emptyset$ for all $y \in E$, then $f_{\mathbb{K}}$ is called a null SS and indicated by Φ_E.*

Definition 2.4. *[40] Let $f_A, f_B \in S_E(U)$. If $f_A(y) \subseteq f_B(y)$ for all $y \in E$, then f_A is a soft subset of f_B and indicated by $f_A \subseteq f_B$. If $f_A(y) = f_B(y)$ for all $y \in E$, then f_A is called soft equal to f_B and denoted by $f_A = f_B$.*

Definition 2.5. *[40] Let $f_A, f_B \in S_E(U)$. The union (intersection) of f_A and f_B is the SS $f_A \widetilde{\cup} f_B$ ($f_A \widetilde{\cap} f_B$), where*

$$(f_A \widetilde{\cup} f_B)(l) = f_A(l) \cup f_B(l), \quad (f_A \widetilde{\cap} f_B)(l) = f_A(l) \cap f_B(l),$$

for all $l \in E$, respectively.

Definition 2.6. *[40] Let $f_A, f_B \in S_E(U)$. Then, $\wedge$-product ($\vee$-product) of f_A and f_B, denoted by $f_A \wedge f_B$ ($f_A \vee f_B$) is defined by*

$$(f_A \wedge f_B)(x, y) = f_A(x) \cap f_B(y), \quad (f_A \vee f_B)(x, y) = f_A(x) \cup f_B(y),$$

for all $(x, y) \in E \times E$, respectively.

Definition 2.7. *[42] Let $f_A, f_B \in S_E(U)$ and φ be a function from A to B. Then, the soft anti image of f_A under φ, and the soft pre-image (or soft inverse image) of f_B under φ are the SS $\varphi(f_A)$ and $\varphi^{-1}(f_B)$ such that*

$$(\varphi(f_A))(x) = \begin{cases} \bigcap\{f_A(k) \mid k \in A \text{ and } \varphi(k) = x\}, & \text{if } \varphi^{-1}(x) \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $b \in B$, and $\left(\varphi^{-1}(f_B)\right)(k) = f_B(\varphi(k))$ for all $k \in A$.

Definition 2.8. *[42] Let $f_A \in S_E(U)$ and $\alpha \subseteq U$. Then, the lower α-inclusion of f_A, denoted by $\mathcal{L}(f_A; \alpha)$, is defined as*

$$\mathcal{L}(f_A; \alpha) = \{x \in E \mid f_A(x) \subseteq \alpha\}.$$

Definition 2.9. *[42] Let $f_{\mathbb{K}}, f_{\mathbb{K}} \in S_S(U)$, where $\mathbb{K}$ is a semigroup. Soft union product $f_{\mathbb{K}} \star f_{\mathbb{K}}$ is defined by*

$$(f_{\mathbb{K}} \star f_{\mathbb{K}})(m) = \begin{cases} \bigcup\limits_{m=xy} \left(f_{\mathbb{K}}(x) \cup f_{\mathbb{K}}(y)\right), & \text{if } \exists x, y \in \mathbb{K} \text{ such that } m = xy \\ U, & \text{otherwise} \end{cases}$$

Theorem 2.10. *[42] Let $f_{\mathbb{K}}, f_{\mathbb{K}}, p_{\mathbb{K}} \in S_{\mathbb{K}}(U)$, where $\mathbb{K}$ is a semigroup. Then,*

(i) $(f_{\mathbb{K}} \star f_{\mathbb{K}}) \star p_{\mathbb{K}} = f_{\mathbb{K}} \star (f_{\mathbb{K}} \star p_{\mathbb{K}})$
(ii) $f_{\mathbb{K}} \star f_{\mathbb{K}} = f_{\mathbb{K}} \star f_{\mathbb{K}}$
(iii) $f_{\mathbb{K}} \star (f_{\mathbb{K}} \cup p_{\mathbb{K}}) = (f_{\mathbb{K}} \star f_{\mathbb{K}}) \cup (f_{\mathbb{K}} \star p_{\mathbb{K}})$ *and* $(f_{\mathbb{K}} \cup f_{\mathbb{K}}) \star p_{\mathbb{K}} = (f_{\mathbb{K}} \star p_{\mathbb{K}}) \cup (f_{\mathbb{K}} \star p_{\mathbb{K}})$
(iv) $f_{\mathbb{K}} \star (f_{\mathbb{K}} \cup p_{\mathbb{K}}) = (f_{\mathbb{K}} \star f_{\mathbb{K}}) \cup (f_{\mathbb{K}} \star p_{\mathbb{K}})$ *and* $(f_{\mathbb{K}} \cup f_{\mathbb{K}}) \star p_{\mathbb{K}} = (f_{\mathbb{K}} \star p_{\mathbb{K}}) \cup (f_{\mathbb{K}} \star p_{\mathbb{K}})$
(v) If $f_{\mathbb{K}} \subseteq t_{\mathbb{K}}$, then $f_{\mathbb{K}} \star p_{\mathbb{K}} \subseteq t_{\mathbb{K}} \star p_{\mathbb{K}}$ and $p_{\mathbb{K}} \star f_{\mathbb{K}} \subseteq p_{\mathbb{K}} \star t_{\mathbb{K}}$
(vi) If $z_{\mathbb{K}}, s_{\mathbb{K}} \in S_{\mathbb{K}}(U)$ such that $z_{\mathbb{K}} \subseteq f_{\mathbb{K}}$ and $s_{\mathbb{K}} \subseteq f_{\mathbb{K}}$, then $z_{\mathbb{K}} \star s_{\mathbb{K}} \subseteq f_{\mathbb{K}} \star f_{\mathbb{K}}$

Definition 2.11. *[42] Let $\emptyset \neq K \subseteq S$. The soft characteristic function (Schf) of the complement K, denoted by S_{K^c}, is defined as*

$$S_{K^c}(x) = \begin{cases} \emptyset, & \text{if } x \in K \\ U, & \text{if } x \in S \setminus K \end{cases}$$

Theorem 2.12. *[42] Let $G, B \subseteq S$. Then*

(i) If $G \subseteq B$, then $S_B \subseteq S_G$
(ii) $S_{G \cup B^c} = S_G \cup S_B^c$ and $S_{G \cap B^c} = S_G \cap S_B^c$

From now on, $\mathbb{K}$ denotes a semigroup likewise S.

Definition 2.13. *[42] An SS $f_{\mathbb{K}}$ over U is called an S-uni subsemigroup of S if $f_{\mathbb{K}}(l\eta) \subseteq f_{\mathbb{K}}(l) \cup f_{\mathbb{K}}(\eta)$ for all $l, \eta \in S$.*

Note that in [42], the definition of "S-uni subsemigroup of S" is given as "S-uni semigroup of S"; however in this paper, without loss of generality, we prefer to use "S-uni subsemigroup of S".

Definition 2.14. *[42, 43] An SS $f_{\mathbb{K}}$ over U is called an S-uni $\mathcal{L}$ (resp. $\mathcal{R}$) ideal of S if $f_{\mathbb{K}}(l\eta) \subseteq f_{\mathbb{K}}(\eta)$ (resp. $f_{\mathbb{K}}(l\eta) \subseteq f_{\mathbb{K}}(l)$) for all $l, \eta \in S$, and is called an S-uni two-sided ideal (S-uni ideal) of S if it is both S-uni $\mathcal{L}$ ideal of S over U and S-uni $\mathcal{R}$ ideal of S over U. An S-uni subsemigroup $f_{\mathbb{K}}$ is called an S-uni bi-ideal of S if $f_{\mathbb{K}}(l\eta t) \subseteq f_{\mathbb{K}}(l) \cup f_{\mathbb{K}}(t)$ for all $l, \eta, t \in S$. An SS $f_{\mathbb{K}}$ over U is called an S-uni interior ideal of S if $f_{\mathbb{K}}(l\eta t) \subseteq f_{\mathbb{K}}(\eta)$ for all $l, \eta, t \in S$. An SS $f_{\mathbb{K}}$ over U is called an S-uni generalized bi-ideal of S if $f_{\mathbb{K}}(l\eta t) \subseteq f_{\mathbb{K}}(l) \cup f_{\mathbb{K}}(t)$ for all $l, \eta, t \in S$.*

An SS $f_{\mathbb{K}}$ over U is called an S-uni $\mathcal{L}$ weak-interior (resp. $\mathcal{R}$ weak-interior) ideal of S if $f_{\mathbb{K}}(l\eta) \subseteq f_{\mathbb{K}}(\eta) \cup f_{\mathbb{K}}(l)$ (resp. $f_{\mathbb{K}}(l\eta) \subseteq f_{\mathbb{K}}(l) \cup f_{\mathbb{K}}(\eta)$) for all $l, \eta \in S$, and is called an S-uni weak-interior ideal of S if it is both S-uni $\mathcal{L}$ weak-interior ideal of S over U and S-uni $\mathcal{R}$ weak-interior ideal of S over U. An SS $f_{\mathbb{K}}$ over U is called an S-uni $\mathcal{L}$ quasi-interior (resp. $\mathcal{R}$ quasi-interior) ideal of S if $f_{\mathbb{K}}(l\eta t\oslash) \subseteq f_{\mathbb{K}}(\eta) \cup f_{\mathbb{K}}(\oslash)$ (resp. $f_{\mathbb{K}}(l\eta t\oslash) \subseteq f_{\mathbb{K}}(l) \cup f_{\mathbb{K}}(t)$) for all $l, \eta, t, \oslash \in S$, and is called an S-uni quasi-interior ideal of S if it is both S-uni $\mathcal{L}$ quasi-interior ideal of S over U and S-uni $\mathcal{R}$ quasi-interior ideal of S over U [75, 76].

If $f_{\mathbb{K}}(x) = U$ for all $x \in S$, then $f_{\mathbb{K}}$ is an S-uni subsemigroup ($\mathcal{L}$ ideal, $\mathcal{R}$ ideal, ideal, bi-ideal, interior ideal, generalized bi-ideal, $\mathcal{L}$ weak-interior ideal, $\mathcal{R}$ weak-interior ideal, weak-interior ideal, $\mathcal{L}$ quasi-interior ideal, $\mathcal{R}$ quasi-interior ideal, quasi-interior ideal). We denote such a kind of S-uni subsemigroup ($\mathcal{L}$ ideal, $\mathcal{R}$ ideal, ideal, bi-ideal, interior ideal, generalized bi-ideal, $\mathcal{L}$ weak-interior ideal, $\mathcal{R}$ weak-interior ideal, weak-interior ideal, $\mathcal{L}$ quasi-interior ideal, $\mathcal{R}$ quasi-interior ideal, quasi-interior ideal) by $\tilde{O}$. Moreover, $\tilde{O} = S_{K^c}$, that is, $\tilde{O}(x) = \emptyset$ for all $x \in S$ [42, 43, 75, 76].

Definition 2.15. *[42, 77, 78] An SS $f_{\mathbb{K}}$ over U is called an S-uni quasi-ideal of S over U if $(\tilde{O} \star f_{\mathbb{K}}) \cup (f_{\mathbb{K}} \star \tilde{O}) \subseteq f_{\mathbb{K}}$. An SS $f_{\mathbb{K}}$ over U is called an S-uni bi-interior ideal of S over U if $(\tilde{O} \star f_{\mathbb{K}} \star \tilde{O}) \cup (f_{\mathbb{K}} \star \tilde{O} \star f_{\mathbb{K}}) \subseteq f_{\mathbb{K}}$. An SS $f_{\mathbb{K}}$ over U is called an S-uni $\mathcal{L}$ bi-quasi (resp. $\mathcal{R}$ bi-quasi) ideal of S if $(\tilde{O} \star f_{\mathbb{K}}) \cup (f_{\mathbb{K}} \star \tilde{O} \star f_{\mathbb{K}}) \subseteq f_{\mathbb{K}}$ (resp. $(f_{\mathbb{K}} \star \tilde{O}) \cup (\tilde{O} \star f_{\mathbb{K}} \star f_{\mathbb{K}}) \subseteq f_{\mathbb{K}}$), and is called an S-uni bi-quasi ideal of S if it is both S-uni $\mathcal{L}$ bi-quasi ideal of S over U and S-uni $\mathcal{R}$ bi-quasi ideal of S over U.*

Theorem 2.16. *[42] Let $f_S \in S_S(U)$. Then,*

(i) $\tilde{O} \star \tilde{O} \subseteq \tilde{O}$
(ii) $\tilde{O} \star f_S \subseteq \tilde{O}$ *and* $f_S \star \tilde{O} \subseteq \tilde{O}$
(iii) $f_S \cap \tilde{O} = \tilde{O}$ *and* $f_S \cup \tilde{O} = f_S$

Theorem 2.17. *[42, 43] Let $\mathbb{K}$ be a nonempty subset of a semigroup S. Then, $\mathbb{K}$ is a subsemigroup ($\mathcal{L}$ ideal, $\mathcal{R}$ ideal, two-sided ideal, bi-ideal, interior ideal, quasi-ideal) of S iff $S_{\mathbb{K}}$ is an S-uni subsemigroup ($\mathcal{L}$ ideal, $\mathcal{R}$ ideal, two-sided ideal, bi-ideal, interior ideal, quasi-ideal).*

Theorem 2.18. *[42, 43, 75, 76] Let $f_S \in S_S(U)$. Then,*

(1) f_S is an S-uni subsemigroup iff $(f_S \star f_S) \subseteq f_S$,
(2) f_S is an S-uni $\mathcal{L}$ (resp. $\mathcal{R}$) ideal iff $(\tilde{O} \star f_S) \subseteq f_S$ and $(f_S \star \tilde{O}) \subseteq f_S$,
(3) f_S is an S-uni bi-ideal iff $(f_S \star f_S) \subseteq f_S$ and $(f_S \star \tilde{O} \star f_S) \subseteq f_S$,
(4) f_S is an S-uni interior ideal iff $(\tilde{O} \star f_S \star \tilde{O}) \subseteq f_S$,
(5) f_S is an S-uni $\mathcal{R}$ weak-interior ideal iff $(\tilde{O} \star f_S \star f_S) \subseteq f_S$ and $((f_S \star f_S \star \tilde{O}) \subseteq f_S)$,
(6) f_S is an S-uni $\mathcal{R}$ quasi-interior ideal iff $(\tilde{O} \star f_S \star \tilde{O} \star f_S) \subseteq f_S$ and $((f_S \star \tilde{O} \star f_S \star \tilde{O}) \subseteq f_S)$

Theorem 2.19. *[42, 43] The following assertions hold:*

(1) Every S-uni $\mathcal{L}$ (resp. $\mathcal{R}$/two-sided) ideal is an S-uni subsemigroup (S-uni bi-ideal/S-uni quasi-ideal).
(2) Every S-uni ideal is an S-uni interior ideal (S-uni quasi-ideal).
(3) Every S-uni quasi-ideal is an S-uni subsemigroup (S-uni bi-ideal).

Theorem 2.20. *[42] Let $f_S \in S_S(U)$, α be a subset of U, $Im(f_S)$ be the image of f_S such that $\alpha \in Im(f_S)$. If f_S is an S-uni subsemigroup of S, then $\mathcal{U}(f_S; \alpha)$ is a subsemigroup of S.*

3. Results and Discussions

In this section, we present the concept of soft union (S-uni) bi-quasi-interior ideals of semigroups, provide its examples, thoroughly examine its relationships with other soft union ideals, and analyze the concept in terms of certain SS concepts and operations.

Definition 3.1. *A soft set f_S over U is called a soft union (S-uni) BQI ideal of S over U if $f_S(b \star d \star m \star n) \subseteq f_S(b) \cup f_S(d) \cup f_S(m) \cup f_S(n)$ for all $b, d, f, m, n \in S$.*

S-uni bi-quasi-interior ideal of S over U is abbreviated by S-uni BQI-ideal in what follows.

Example 3.2. *Let $S = \{z, \sqcup, \mathfrak{u}\}$ be defined by the following table:*

$\star$	z	$\sqcup$	$\mathfrak{u}$
z	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\sqcup$	z	$\sqcup$	$\mathfrak{u}$
$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$

Let $\mathfrak{t}_S$ and $\mathfrak{q}_S$ be soft sets over $U = \mathbb{Z}_8^*$ as follows:
$\mathfrak{t}_S = \{(z, \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}), (\sqcup, \{\overline{1}, \overline{3}, \overline{7}\}), (\mathfrak{u}, \{\overline{1}, \overline{3}\})\}$, $\mathfrak{q}_S = \{(z, \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}), (\sqcup, \{\overline{1}, \overline{3}, \overline{7}\}), (\mathfrak{u}, \{\overline{1}, \overline{3}, \overline{5}\})\}$.
$\mathfrak{t}_S$ is an S-uni BQI-ideal of S. Here, we find it appropriate to give a few concrete examples of elements for ease of illustration in order to be more understandable. In fact,

$$\mathfrak{t}_S(zzz \sqcup \mathfrak{u}\mathfrak{u}) = \mathfrak{t}_S(\mathfrak{u}) \subseteq \mathfrak{t}_S(z) \cup \mathfrak{t}_S(\sqcup) \cup \mathfrak{t}_S(\mathfrak{u}),$$
$$\mathfrak{t}_S(\sqcup \sqcup \sqcup \mathfrak{u}\mathfrak{u}\mathfrak{u}) = \mathfrak{t}_S(\mathfrak{u}) \subseteq \mathfrak{t}_S(\sqcup) \cup \mathfrak{t}_S(\sqcup) \cup \mathfrak{t}_S(\mathfrak{u}),$$
$$\mathfrak{t}_S(zzz \sqcup \sqcup z) = \mathfrak{t}_S(\mathfrak{u}) \subseteq \mathfrak{t}_S(z) \cup \mathfrak{t}_S(\sqcup) \cup \mathfrak{t}_S(z).$$

It can be easily shown that the soft set $\mathfrak{t}_S$ satisfies the S-uni BQI-ideal condition for all other element combinations of the set S. However, since $\mathfrak{q}_S(\sqcup \sqcup \sqcup \mathfrak{u}\mathfrak{u}\sqcup) = \mathfrak{q}_S(\mathfrak{u}) \not\subseteq \mathfrak{q}_S(\sqcup) \cup \mathfrak{q}_S(\mathfrak{u}) \cup \mathfrak{q}_S(\sqcup)$, $\mathfrak{q}_S$ is not an S-uni BQI-ideal.

Theorem 3.3. *Let $f_S \in S_S(U)$. Then, f_S is an S-uni BQI-ideal if and only if $f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S \subseteq f_S$.*

Proof. Suppose that f_S is an S-uni BQI-ideal and $s \in S$. If $(f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S)(s) = \emptyset$, then $f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S \subseteq f_S$. Otherwise, there exist elements $l, n, t, \oslash, z, \sqcup, \mathfrak{u}, b \in S$ such that $s = ln$, $b = t\oslash$, $t = z\sqcup$, and $z = \mathfrak{u}b$, for $s \in S$.

Since f_S is an S-uni BQI-ideal,

$$f_S(s) = f_S(ln) = f_S(\mathfrak{u}b \sqcup n) \subseteq f_S(\mathfrak{u}) \cup f_S(\sqcup) \cup f_S(n).$$

Therefore,

$$\begin{aligned}
(f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S)(s) &= [(f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S)](s) \\
&= \bigcap_{s=ln} \left\{ (f_S \star \tilde{O} \star f_S \star \tilde{O})(l) \cup f_S(n) \right\} \\
&= \bigcap_{s=ln} \left\{ \bigcap_{l=tb} \left((f_S \star \tilde{O})(t) \cup \tilde{O}(b) \right) \cup f_S(n) \right\} \\
&= \bigcap_{\substack{s=ln \\ l=tb}} \left\{ \bigcap_{t=z\sqcup} \left((f_S \star \tilde{O})(z) \cup \tilde{O}(\sqcup) \right) \cup \tilde{O}(b) \cup f_S(n) \right\} \\
&= \bigcap_{\substack{s=ln \\ l=tb \\ t=z\sqcup}} \left\{ \bigcap_{z=\mathfrak{u}b} \left(f_S(\mathfrak{u}) \cup \tilde{O}(b) \right) \cup f_S(\sqcup) \cup \tilde{O}(\oslash) \cup f_S(n) \right\} \\
&= \bigcap_{s=\mathfrak{u}b\sqcup n} \{ f_S(\mathfrak{u}) \cup f_S(\sqcup) \cup f_S(n) \} \\
&\subseteq \bigcap_{s=\mathfrak{u}b\sqcup n} f_S(\mathfrak{u}b \sqcup n) \\
&= f_S(ln) \\
&= f_S(s).
\end{aligned}$$

□

Thus, we have $f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S \subseteq f_S$. Moreover, in the case where $s = ln$ and $b \neq \mathfrak{u}b \sqcup n$ for $s \in S$, since $(f_S \star \tilde{O} \star f_S \star \tilde{O})(b) = \emptyset$, $f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S \subseteq f_S$ is satisfied.

Conversely, assume that $f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S \subseteq f_S$. Let $s = l \oslash t \oslash z$ for $l, t, \oslash, z \in S$. Then, we have:

$$\begin{aligned}
f_S(l \oslash t \oslash z) &= f_S(s) \\
&\subseteq (f_S \star \tilde{O} \star f_S \star \tilde{O} \star f_S)(s) \\
&= [(f_S \star \tilde{O} \star f_S \star \tilde{O}) \star f_S](s) \\
&= \bigcap_{s=bz} \left\{ (f_S \star \tilde{O} \star f_S \star \tilde{O})(b) \cup f_S(z) \right\} \\
&= \bigcap_{s=bz} \left\{ \bigcap_{b=\mathfrak{u}\oslash} \Big((f_S \star \tilde{O} \star f_S)(\mathfrak{u}) \cup \tilde{O}(\oslash) \cup f_S(z) \Big) \right\} \\
&= \bigcap_{s=bz} \bigcap_{b=\mathfrak{u}\oslash} \bigcap_{\mathfrak{u}=ln} \Big((f_S \star \tilde{O})(l) \cup f_S(t) \cup \tilde{O}(\oslash) \cup f_S(z) \Big) \\
&= \bigcap_{s=bz} \bigcap_{b=\mathfrak{u}\oslash} \bigcap_{\mathfrak{u}=ln} \bigcap_{l=bt} \Big(f_S(l) \cup \tilde{O}(\oslash) \cup f_S(t) \cup \tilde{O}(\oslash) \cup f_S(z) \Big) \\
&\subseteq (f_S \star \tilde{O} \star f_S \star \tilde{O})(l \oslash t \oslash) \cup f_S(z) \\
&= \bigcap_{s=l \oslash t \oslash z} \Big(f_S(l) \cup \tilde{O}(\oslash) \cup f_S(t) \cup \tilde{O}(\oslash) \cup f_S(z) \Big) \\
&= f_S(l) \cup \tilde{O}(\oslash) \cup f_S(t) \cup \tilde{O}(\oslash) \cup f_S(z) \\
&= f_S(l) \cup f_S(t) \cup f_S(z).
\end{aligned}$$

Hence, $f_S(l \oslash t \oslash z) \subseteq f_S(l) \cup f_S(t) \cup f_S(z)$ implying that f_S is an S-uni BQI-ideal.

Corollary 3.4. *$\tilde{O}$ is an S-uni BQI-ideal.*

Proposition 3.5. *$\emptyset \neq \mathbb{K} \subseteq S$ is a BQI ideal if the S-uni subsemigroup f_S defined by*

$$f_S(\varsigma) = \begin{cases} \alpha, & \text{if } \varsigma \in S \setminus \mathbb{K} \\ \beta, & \text{if } \varsigma \in \mathbb{K} \end{cases}$$

is an S-uni BQI ideal, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof. Let $\mathbb{K}$ be a *BQI* ideal and $\mathfrak{x}, \mathfrak{u}, b, z \in S$. If $\mathfrak{x}, \mathfrak{u}, z \in \mathbb{K}$, then $\mathfrak{x}\mathfrak{u}bz \in \mathbb{K}$. Hence, $f_S(\mathfrak{x}\mathfrak{u}bz) = f_S(\mathfrak{x}) = f_S(\mathfrak{u}) = f_S(z) = \beta$ and so $f_S(\mathfrak{x}\mathfrak{u}bz) \subseteq f_S(\mathfrak{x}) \cup f_S(\mathfrak{u}) \cup f_S(z)$. If $\mathfrak{x} \notin \mathbb{K}$, $\mathfrak{u} \notin \mathbb{K}$ and $z \notin \mathbb{K}$, then $\mathfrak{x}\mathfrak{u}bz \notin \mathbb{K}$ or $\mathfrak{x}\mathfrak{u}bz \in \mathbb{K}$. In this case, if $\mathfrak{x}\mathfrak{u}bz \in \mathbb{K}$, then $\beta = f_S(\mathfrak{x}\mathfrak{u}bz) = f_S(\mathfrak{x}) \cup f_S(\mathfrak{u}) \cup f_S(z) = \alpha$. If $\mathfrak{x}\mathfrak{u}bz \notin \mathbb{K}$, then $\alpha = f_S(\mathfrak{x}\mathfrak{u}bz) \subseteq f_S(\mathfrak{x}) \cup f_S(\mathfrak{u}) \cup f_S(z) = \alpha$. If $\mathfrak{x} \in \mathbb{K}$ or $\mathfrak{u} \in \mathbb{K}$ or $z \in \mathbb{K}$, then $\mathfrak{x}\mathfrak{u}bz \in \mathbb{K}$ or $\mathfrak{x}\mathfrak{u}bz \notin \mathbb{K}$. Here, firstly note that, if $\mathfrak{x} \in \mathbb{K}$, $\mathfrak{u} \in \mathbb{K}$, or $z \in \mathbb{K}$, then either $f_S(\mathfrak{x}) \cup f_S(\mathfrak{u}) \cup f_S(z) = \beta$ (the case where $\mathfrak{x} \in \mathbb{K}$, $\mathfrak{u} \in \mathbb{K}$, and $z \in \mathbb{K}$), or $f_S(\mathfrak{x}) \cup f_S(\mathfrak{u}) \cup f_S(z) = \alpha$ (the case where $\mathfrak{x} \in \mathbb{K}$ and $\mathfrak{u} \notin \mathbb{K}$ or $z \notin \mathbb{K}$, or $\mathfrak{u} \notin \mathbb{K}$ and $\mathfrak{x} \notin \mathbb{K}$ or $z \notin \mathbb{K}$, and $\mathfrak{x} \notin \mathbb{K}$ or $\mathfrak{u} \notin \mathbb{K}$). Thus, either $\mathfrak{x}\mathfrak{u}bz \in \mathbb{K}$ or $\mathfrak{x}\mathfrak{u}bz \notin \mathbb{K}$. However, in any case $f_S(\mathfrak{x}\mathfrak{u}bz) \subseteq f_S(\mathfrak{x}) \cup f_S(\mathfrak{u}) \cup f_S(z)$, since $\alpha \supseteq \beta$. Hence, f_S is an S-uni *BQI* ideal.

Conversely assume that S-uni subsemigroup f_S is an S-uni *BQI* ideal. Let $\mathfrak{x}, \mathfrak{u} \in \mathbb{K}$ and $b, z \in S$. Then, $f_S(\mathfrak{x}\mathfrak{u}bz) \subseteq f_S(\mathfrak{x}) = f_S(\mathfrak{u}) = f_S(z) = \beta$. Since $\beta \subseteq \alpha$ and the function is two-valued, $f_S(\mathfrak{x}\mathfrak{u}bz) \neq \alpha$, implying that $f_S(\mathfrak{x}\mathfrak{u}bz) = \beta$. Hence, $\mathfrak{x}\mathfrak{u}bz \in \mathbb{K}$ and $\mathbb{K}$ is a *BQI* ideal. □

Theorem 3.6. *Let $\mathbb{K}$ be a subsemigroup of S. Then, $\mathbb{K}$ is a BQI ideal of S iff $S_{\mathbb{K}^c}$ is an S-uni BQI ideal.*

Proof. Since

$$S_{\mathbb{K}^c}(\varsigma) = \begin{cases} U, & \text{if } \varsigma \in S \setminus \mathbb{K} \\ \emptyset, & \text{if } \varsigma \in \mathbb{K} \end{cases}$$

and $U \supseteq \emptyset$, the remainder of the proof is completed based on Proposition 3.5. □

Example 3.7. *We consider the semigroup in Example 3.2. $A = \{z, \mathfrak{u}\}$ is a BQI ideal of S. By the definition $\mathcal{S}_{c\mathfrak{C}}$, $S_A = \{(z, \emptyset), (\sqcup, U), (\mathfrak{u}, \emptyset)\}$. S_A is an S-uni BQI-ideal. Conversely, by choosing the S-uni BQI-ideal as $\mathfrak{t}_S = \{(z, \emptyset), (\sqcup, U), (\mathfrak{u}, \emptyset)\}$, which is the $\mathcal{S}_{c\mathfrak{C}}$ of the complement of $X = \{z, \mathfrak{u}\}$, X is a BQI ideal of S.*

Now, we continue with the relationships between S-uni BQI-ideals and other types of S-uni ideals of S.

Theorem 3.8. *Every S-uni bi-ideal is an S-uni BQI ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni bi-ideal of S. Then, $\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S \subseteq \mathfrak{t}_S$. Thus, $(\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S) \star \tilde{O} \star \mathfrak{t}_S \subseteq \mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S \subseteq \mathfrak{t}_S$. Hence, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. □

We show with a counterexample that the converse of Theorem 3.8. is not true:

Example 3.9. ***3.9.*** *Let $S = \{\mathfrak{u}, \emptyset, \mathfrak{b}, \mathfrak{w}\}$ be defined by the following table:*

$\otimes$	$\mathfrak{u}$	$\emptyset$	$\mathfrak{b}$	$\mathfrak{w}$
$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\emptyset$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{b}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\emptyset$
$\mathfrak{w}$	$\mathfrak{u}$	$\emptyset$	$\emptyset$	$\mathfrak{b}$

Let $\mathfrak{t}_S$ be a soft set over $U = \mathbb{Z}$ as follows: $\mathfrak{t}_S = \{(\mathfrak{u}, \{1,2\}), (\emptyset, \{1,2,3,4,5\}), (\mathfrak{b}, \{1,2,5\}), (\mathfrak{w}, \{1,2,3\})\}$. Here, $\mathfrak{t}_S$ is an S-uni BQI-ideal. In fact,

$$(\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S)(\mathfrak{u}) = \mathfrak{t}_S(\mathfrak{u}) \cap \mathfrak{t}_S(\emptyset) \cap \mathfrak{t}_S(\mathfrak{b}) \cap \mathfrak{t}_S(\mathfrak{w}) \supseteq \mathfrak{t}_S(\mathfrak{u}),$$

$$(\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S)(\emptyset) = U \supseteq \mathfrak{t}_S(\emptyset),$$

$$(\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S)(\mathfrak{b}) = U \supseteq \mathfrak{t}_S(\mathfrak{b}),$$

$$(\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S)(\mathfrak{w}) = U \supseteq \mathfrak{t}_S(\mathfrak{w}).$$

Thus, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. However, since $(\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S)(\emptyset) = \mathfrak{t}_S(\mathfrak{w}) \neq \mathfrak{t}_S(\emptyset)$, $\mathfrak{t}_S$ is not an S-uni bi-ideal.

Theorem 3.10. *Let $\mathfrak{t}_S \in S_S(U)$ and S be an $\mathcal{R}$ semigroup. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ is an S-uni bi-ideal.
(2) $\mathfrak{t}_S$ is an S-uni BQI-ideal.

Proof. (1) implies (2) is obvious by Theorem 3.8. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal and $b, n, t \in S$. By assumption, there exist $m, r \in S$ such that $b = lmr$ and $n = nrn$. Thus,

$$\mathfrak{t}_S(lnb) = \mathfrak{t}_S((lmr)nrb) = \mathfrak{t}_S(lmrlmrn) \subseteq \mathfrak{t}_S(l) \cup \mathfrak{t}_S(l) \cup \mathfrak{t}_S(t) = \mathfrak{t}_S(l) \cup \mathfrak{t}_S(t),$$

$$\mathfrak{t}_S(ln) = \mathfrak{t}_S(lmrl(nrn)) = \mathfrak{t}_S(l) \cup \mathfrak{t}_S(n) = \mathfrak{t}_S(l) \cup \mathfrak{t}_S(n).$$

Thus, $\mathfrak{t}_S$ is an S-uni bi-ideal. □

Proposition 3.11. *Every S-uni $\mathcal{L}$ ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni $\mathcal{L}$ ideal of S. Then, by Theorem 2.19, $\mathfrak{t}_S$ is an S-uni bi-ideal. The rest of the proof is obvious by Theorem 3.8. Hence, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. □

We show with a counterexample that the converse of Proposition 3.11 is not true:

Example 3.12. *Let the semigroup* $\mathfrak{R} = \{\mathfrak{d}, \mathfrak{k}\}$ *be defined by the following table:*

$\star$	$\mathfrak{d}$	$\mathfrak{k}$
$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{d}$
$\mathfrak{k}$	$\mathfrak{k}$	$\mathfrak{k}$

Let $\mathfrak{t}_{\mathfrak{R}}$ *be a soft set over* $U = \{\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \mid x \in \mathbb{Z}\}$ *as follows:* $\mathfrak{t}_{\mathfrak{R}} = \{(\mathfrak{d}, \{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right]\}), (\mathfrak{k}, \{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]\})\}$. *Here,* $\mathfrak{t}_{\mathfrak{R}}$ *is an S-uni BQI-ideal. In fact,*

$$(\mathfrak{t}_{\mathfrak{R}} \star \tilde{O} \star \mathfrak{t}_{\mathfrak{R}} \star \tilde{O} \star \mathfrak{t}_{\mathfrak{R}})(\mathfrak{d}) = \mathfrak{t}_{\mathfrak{R}}(\mathfrak{d}) \supseteq \mathfrak{t}_{\mathfrak{R}}(\mathfrak{d}),$$

$$(\mathfrak{t}_{\mathfrak{R}} \star \tilde{O} \star \mathfrak{t}_{\mathfrak{R}} \star \tilde{O} \star \mathfrak{t}_{\mathfrak{R}})(\mathfrak{k}) = \mathfrak{t}_{\mathfrak{R}}(\mathfrak{k}) \supseteq \mathfrak{t}_{\mathfrak{R}}(\mathfrak{k}).$$

Thus, $\mathfrak{t}_{\mathfrak{R}}$ *is an S-uni BQI-ideal of* $\mathfrak{R}$. *However, since* $\mathfrak{t}_{\mathfrak{R}}(\mathfrak{k}\mathfrak{d}) = \mathfrak{t}_{\mathfrak{R}}(\mathfrak{k}) \neq \mathfrak{t}_{\mathfrak{R}}(\mathfrak{d})$, $\mathfrak{t}_{\mathfrak{R}}$ *is not an S-uni* $\mathcal{L}$ *ideal.*

Proposition 3.13. *Let* $\mathfrak{t}_S \in S_S(U)$ *and* S *be an* $\mathcal{R}$ *semigroup and* $\mathcal{R}$ *simple semigroup. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni* $\mathcal{L}$ *ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. (1) implies (2) is obvious by Proposition 3.11. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal and $b, n, t \in S$. By assumption, there exist $m, r \in S$ such that $b = nmr$ and $n = nrn$. Thus, $\mathfrak{t}_S(ln) = \mathfrak{t}_S((nmr)(nrn)) = \mathfrak{t}_S(nmrnrn) \subseteq \mathfrak{t}_S(n) \cup \mathfrak{t}_S(n) \cup \mathfrak{t}_S(n) = \mathfrak{t}_S(n)$. Thus, $\mathfrak{t}_S$ is an S-uni $\mathcal{L}$ ideal. □

Proposition 3.14. *Every S-uni* $\mathcal{R}$ *ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni $\mathcal{R}$ ideal of S. Then, by Theorem 2.19, $\mathfrak{t}_S$ is an S-uni bi-ideal. The rest of the proof is obvious by Theorem 3.8. Hence, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. □

We show with a counterexample that the converse of Proposition 3.14 is not true:

Example 3.15. *Consider the SS* $\mathfrak{t}_S$ *in Example 3.2. It was shown in Example 3.2 that* $\mathfrak{t}_S$ *is an S-uni BQI-ideal. Since* $\mathfrak{t}_S(luz) = \mathfrak{t}_S(z) \neq \mathfrak{t}_S(lu)$, $\mathfrak{t}_S$ *is not an S-uni* $\mathcal{R}$ *ideal.*

Proposition 3.16. *Let* $\mathfrak{t}_S \in S_S(U)$ *and* S *be an* $\mathcal{R}$ *semigroup and* $\mathcal{L}$ *simple semigroup. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni* $\mathcal{R}$ *ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. (1) implies (2) is obvious by Proposition 3.14. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal and $l, n \in S$. By assumption, there exist $m, r \in S$ such that $n = mr$ and $l = lml$.
Thus, $\mathfrak{t}_S(ln) = \mathfrak{t}_S((lml)(mr)) = \mathfrak{t}_S(lmlmr) \subseteq \mathfrak{t}_S(l) \cup \mathfrak{t}_S(l) \cup \mathfrak{t}_S(l) = \mathfrak{t}_S(l)$. Thus, $\mathfrak{t}_S$ is an S-uni $\mathcal{R}$ ideal. □

Theorem 3.17. *Every S-uni ideal is an S-uni BQI-ideal.*

Proof. It is followed by Proposition 3.14 and Proposition 3.16. □

Here note that the converse of Theorem 3.17 is not true follows from Example 3.12 and Example 3.15.
Theorem 3.18 shows that the converse of Theorem 3.17 holds for $\mathcal{R}$ groups.

Theorem 3.18. *Let* $\mathfrak{t}_S \in S_S(U)$ *and* S *be an* $\mathcal{R}$ *group. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. (1) implies (2) is obvious by Proposition 3.11 and Proposition 3.13. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal of a group S. Then, by Theorem 2.1, S is both an $\mathcal{L}$ simple and an $\mathcal{R}$ simple semigroup. The rest of the proof follows from Proposition 3.11 and Proposition 3.14. □

Theorem 3.19. *Every S-uni interior ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni interior ideal of S. Then, $\tilde{O}\star\mathfrak{t}_S\star\tilde{O} \subseteq \mathfrak{t}_S$. Thus, $\mathfrak{t}_S\star(\tilde{O}\star\mathfrak{t}_S\star\tilde{O})\star\mathfrak{t}_S \subseteq \mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S \star \tilde{O} \subseteq \mathfrak{t}_S$.
Hence, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. □

We show with a counterexample that the converse of Theorem 3.19 is not true:

Example 3.20. *Consider the SS $\mathfrak{t}_S$ in Example 3.2. It was shown in Example 3.2 that $\mathfrak{t}_S$ is an S-uni BQI-ideal. Since $\mathfrak{t}_S(luz) = \mathfrak{t}_S(z) \neq \mathfrak{t}_S(lu)$, $\mathfrak{t}_S$ is not an S-uni interior ideal.*

Theorem 3.21 shows that the converse of Theorem 3.19 holds for the groups.

Theorem 3.21. *Let $\mathfrak{t}_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ is an S-uni interior ideal.
(2) $\mathfrak{t}_S$ is an S-uni BQI-ideal.

Proof. (1) implies (2) is obvious by Theorem 3.19. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal and $l, n, t \in S$. By assumption, there exists $m \in S$ such that $l = nmn$ and $t = nn$. Thus, $\mathfrak{t}_S(lnt) = \mathfrak{t}_S((nmn)nt) = \mathfrak{t}_S(nmnnt) \subseteq \mathfrak{t}_S(n) \cup \mathfrak{t}_S(n) \cup \mathfrak{t}_S(n) = \mathfrak{t}_S(n)$. Thus, $\mathfrak{t}_S$ is an S-uni interior ideal. □

Theorem 3.22. *Every S-uni quasi-ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni quasi-ideal of S. Then, by Theorem 2.19, $\mathfrak{t}_S$ is an S-uni bi-ideal. The rest of the proof is obvious by Theorem 3.8. Hence, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. □

We show with a counterexample that the converse of Theorem 3.22 is not true:

Example 3.23. *Consider the SS $\mathfrak{t}_S$ in Example 3.9. It was shown in Example 3.9 that $\mathfrak{t}_S$ is an S-uni BQI-ideal. Since $(\mathfrak{t}_S\star\tilde{O})(\emptyset)\cup(\tilde{O}\star\mathfrak{t}_S)(\emptyset) = \mathfrak{t}_S(\mathfrak{w}) \neq \mathfrak{t}_S(\emptyset)$, $\mathfrak{t}_S$ is not an S-uni quasi-ideal.*

Theorem 3.24. *Let $\mathfrak{t}_S \in S_S(U)$ and S be an $\mathcal{R}$ group. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ is an S-uni quasi-ideal.
(2) $\mathfrak{t}_S$ is an S-uni BQI-ideal.

Proof. (1) implies (2) is obvious by Theorem 3.22. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal. Since S is an $\mathcal{R}$ group, then, by Theorem 3.18, $\mathfrak{t}_S$ is an S-uni ideal. The rest of the proof is obvious by Theorem 2.19. $\mathfrak{t}_S$ is an S-uni quasi-ideal of S. □

Theorem 3.25. *Every S-uni bi-interior ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni bi-interior ideal of S. Then, $(\tilde{O} \star \mathfrak{t}_S \star \tilde{O}) \cup (\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S) \subseteq \mathfrak{t}_S$. Since, $(\mathfrak{t}_S \star \tilde{O}) \star \mathfrak{t}_S \subseteq \tilde{O} \star \mathfrak{t}_S \star \tilde{O}$ and $\mathfrak{t}_S \star (\tilde{O} \star \mathfrak{t}_S) \star \mathfrak{t}_S \subseteq \mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S$, it is obtained that $\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S \subseteq (\tilde{O}\star\mathfrak{t}_S\star\tilde{O})\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S) \subseteq \mathfrak{t}_S$. Hence, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. □

Theorem 3.26. *Let $\mathfrak{t}_S \in S_S(U)$ and S be an $\mathcal{R}$ group. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ is an S-uni bi-interior ideal.
(2) $\mathfrak{t}_S$ is an S-uni BQI-ideal.

Proof. (1) implies (2) is obvious by Theorem 3.25. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal. Since S is an $\mathcal{R}$ group, then, by Theorem 3.10, $\mathfrak{t}_S$ is an S-uni bi-ideal. The rest of the proof is obvious by Theorem 2.19. $\mathfrak{t}_S$ is an S-uni bi-interior ideal of S. □

Proposition 3.27. *Every S-uni $\mathcal{L}$ bi-quasi ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni BQI-ideal of S. Then, $(\tilde{O}\star\mathfrak{t}_S)\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$. Since, $(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S\star\tilde{O})\subseteq\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S$ and $\mathfrak{t}_S\star(\tilde{O}\star\mathfrak{t}_S\star\tilde{O})\star\mathfrak{t}_S\subseteq\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S$, it is obtained that $\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S\subseteq(\tilde{O}\star\mathfrak{t}_S)\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$. Hence, $\mathfrak{t}_S$ is an S-uni $\mathcal{L}$ bi-quasi ideal of S. □

We show with a counterexample that the converse of Proposition 3.27 is not true:

Example 3.28. *Consider the SS $\mathfrak{t}_S$ in Example 3.9. It was shown in Example 3.9 that $\mathfrak{t}_S$ is an S-uni BQI-ideal. Since, $(\mathfrak{t}_S\star\tilde{O})(\emptyset)\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)(\emptyset)=\mathfrak{t}_S(\mathfrak{w})\neq\mathfrak{t}_S(\emptyset)$, $\mathfrak{t}_S$ is not an S-uni $\mathcal{L}$ bi-quasi ideal.*

Proposition 3.29. *Let $\mathfrak{t}_S\in S_S(U)$ and S be an $\mathcal{R}$ simple $\mathcal{R}$ semigroup. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni $\mathcal{L}$ bi-quasi ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. (1) implies (2) is obvious by Proposition 3.27. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal. Since S is $\mathcal{R}$ simple $\mathcal{R}$ semigroup, then, by Theorem 3.10, $\mathfrak{t}_S$ is an S-uni bi-ideal and by Proposition 3.11, $\mathfrak{t}_S$ is an S-uni $\mathcal{L}$ ideal. Since, $(\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$ and $(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$ it is obtained $(\tilde{O}\star\mathfrak{t}_S)\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$. □

Proposition 3.30. *Every S-uni $\mathcal{R}$ bi-quasi ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni BQI-ideal of S. Then, $(\mathfrak{t}_S\star\tilde{O})\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$. Since, $\mathfrak{t}_S\star(\tilde{O}\star\mathfrak{t}_S\star\tilde{O})\subseteq\mathfrak{t}_S\star\tilde{O}$ and $\mathfrak{t}_S\star(\tilde{O}\star\mathfrak{t}_S\star\tilde{O})\star\mathfrak{t}_S\subseteq\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S$, it is obtained that $\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S\subseteq(\mathfrak{t}_S\star\tilde{O})\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$. Hence, $\mathfrak{t}_S$ is an S-uni $\mathcal{R}$ bi-quasi ideal of S. □

We show with a counterexample that the converse of Proposition 3.30 is not true:

Example 3.31. *Consider the SS $\mathfrak{t}_S$ in Example 3.9. It was shown in Example 3.9 that $\mathfrak{t}_S$ is an S-uni BQI-ideal. Since, $(\mathfrak{t}_S\star\tilde{O})(\emptyset)\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)(\emptyset)=\mathfrak{t}_S(\mathfrak{w})\neq\mathfrak{t}_S(\emptyset)$, $\mathfrak{t}_S$ is not an S-uni $\mathcal{R}$ bi-quasi ideal.*

Proposition 3.32. *Let $\mathfrak{t}_S\in S_S(U)$ and S be an $\mathcal{L}$ simple $\mathcal{R}$ semigroup. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni $\mathcal{R}$ bi-quasi ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. (1) implies (2) is obvious by Proposition 3.30. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal. Since S is an $\mathcal{L}$ simple $\mathcal{R}$ semigroup, then, by Theorem 3.10, $\mathfrak{t}_S$ is an S-uni bi-ideal and by Proposition 3.16, $\mathfrak{t}_S$ is an S-uni $\mathcal{R}$ ideal. Since, $(\mathfrak{t}_S\star\tilde{O})\subseteq\mathfrak{t}_S$ and $(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$ it is obtained $(\mathfrak{t}_S\star\tilde{O})\cup(\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S)\subseteq\mathfrak{t}_S$. □

Proposition 3.33. *Every S-uni bi-quasi ideal is an S-uni BQI-ideal.*

Proof. It follows by Proposition 3.27 and Proposition 3.32. □

Theorem 3.34. *Let $\mathfrak{t}_S\in S_S(U)$ and S be an $\mathcal{R}$ group. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni bi-quasi ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Note that the converse of Proposition 3.33 is not true, following from Example 3.28 and Example 3.31. Theorem 3.34 shows that the converse of Proposition 3.33 holds for $\mathcal{R}$ groups.

Proposition 3.35. *Every S-uni $\mathcal{L}$ quasi-interior ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni $\mathcal{L}$ quasi-interior ideal of S. Then, $\tilde{O}\star\mathfrak{t}_S\star\tilde{O}\star\mathfrak{t}_S\subseteq\mathfrak{t}_S$. Thus, $(\mathfrak{t}_S\star\tilde{O})\star\mathfrak{t}_S\star\tilde{O}\subseteq\tilde{O}\star\mathfrak{t}_S\star\tilde{O}\subseteq\mathfrak{t}_S$. Hence, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. □

Proposition 3.36. *Let $\mathfrak{t}_S \in S_S(U)$ and S be an $\mathcal{R}$ simple semigroup. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni $\mathcal{L}$ quasi-interior ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. (1) implies (2) is obvious by Proposition 3.35. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal and $l, n, t, d \in S$. By assumption, there exists $m \in S$ such that $l = n\mathfrak{m}$. Thus,$\mathfrak{t}_S(lntd) = \mathfrak{t}_S((n\mathfrak{m})ntd) = \mathfrak{t}_S(n\mathfrak{m}ntd) \subseteq \mathfrak{t}_S(n) \cup \mathfrak{t}_S(n) \cup \mathfrak{t}_S(d) = \mathfrak{t}_S(n) \cup \mathfrak{t}_S(d)$ Thus, $\mathfrak{t}_S$ is an S-uni $\mathcal{L}$ quasi-interior ideal. □

Proposition 3.37. *Every S-uni $\mathcal{R}$ quasi-interior ideal is an S-uni BQI-ideal.*

Proof. Let $\mathfrak{t}_S$ be an S-uni $\mathcal{R}$ quasi-interior ideal of S. Then, $\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S \star \tilde{O} \subseteq \mathfrak{t}_S$. Thus, $\mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S \star (\tilde{O} \star \mathfrak{t}_S) \subseteq \mathfrak{t}_S \star \tilde{O} \star \mathfrak{t}_S \star \tilde{O} \subseteq \mathfrak{t}_S$. Hence, $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. □

We show with a counterexample that the converse of Proposition 3.37 is not true:

Example 3.38. *Let $S = \{\mathbb{N}, \varphi, \mathfrak{r}, 3, \mathfrak{u}\}$ be defined by the following table:*

$*$	$\mathbb{N}$	φ	$\mathfrak{r}$	3	$\mathfrak{u}$
$\mathbb{N}$	3	3	$\mathbb{N}$	3	3
φ	φ	$\mathbb{N}$	φ	3	3
$\mathfrak{r}$	$\mathbb{N}$	3	$\mathfrak{r}$	3	$\mathfrak{u}$
3	$\mathbb{N}$	3	$\mathfrak{r}$	3	3
$\mathfrak{u}$	$\mathbb{N}$	3	$\mathfrak{r}$	3	$\mathfrak{u}$

Let f_S be a soft set (SS) over U as follows: $f_S = \{(\mathbb{N}, \emptyset), (\varphi, U), (\mathfrak{r}, \emptyset), (3, U), (\mathfrak{u}, U)\}$.Here, f_S is an S-uni BQI-ideal. However, since $f_S(\varphi * \varphi) = f_S(3) \nsubseteq f_S(\varphi)$, f_S is not an S-uni $\mathcal{R}$ quasi-interior ideal.

Proposition 3.39. *Let $\mathfrak{t}_S \in S_S(U)$ and S be an $\mathcal{L}$ simple semigroup. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni $\mathcal{R}$ quasi-interior ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. (1) implies (2) is obvious by Proposition 3.37. Assume that $\mathfrak{t}_S$ is an S-uni BQI-ideal and $l, n, t, d \in S$. By assumption, there exists $m \in S$ such that $d = m\mathfrak{t}$. Thus,

$$\mathfrak{t}_S(lntd) = \mathfrak{t}_S(lnt(m\mathfrak{t})) \subseteq \mathfrak{t}_S(l) \cup \mathfrak{t}_S(t) \cup \mathfrak{t}_S(t) = \mathfrak{t}_S(l) \cup \mathfrak{t}_S(t)$$

Thus, $\mathfrak{t}_S$ is an S-uni $\mathcal{R}$ quasi-interior ideal. □

Theorem 3.40. *Every S-uni quasi-interior ideal is an S-uni BQI-ideal.*

Proof. It follows from Proposition 3.35 and Proposition 3.37. □

Theorem 3.41. *Let $\mathfrak{t}_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:*

(1) $\mathfrak{t}_S$ *is an S-uni quasi-interior ideal.*
(2) $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. (1) implies (2) is obvious by Proposition 3.35 and Proposition 3.37. Let S be a group. The rest of the proof is obvious by Proposition 3.36 and Proposition 3.39. □

Note here that the converse of Theorem 3.40 is not true, following from Example 3.43. Theorem 3.41 shows that the converse of Theorem 3.40 holds for groups.

Theorem 3.42. *Let $\mathfrak{t}_S$ and $\mathfrak{t}_T$ be S-uni BQI-ideals of S and T, respectively. Then, $\mathfrak{t}_S \vee \mathfrak{t}_T$ is an S-uni BQI-ideal of $S \times T$.*

Proof. Let $(s_1,t_1),(s_2,t_2),(s_3,t_3),(s_4,t_4),(s_5,t_5)\in S\times T$. Then,

$$\begin{aligned}\delta_{S\vee T}((s_1,t_1)(s_2,t_2)(s_3,t_3)(s_4,t_4)(s_5,t_5)) &= \delta_{S\vee T}(s_1s_2s_3s_4s_5, t_1t_2t_3t_4t_5)\\ &= \delta_S(s_1s_2s_3s_4s_5)\cup\delta_T(t_1t_2t_3t_4t_5)\\ &\subseteq (\delta_S(s_1)\cup\delta_S(s_2))\cup(\delta_T(t_3)\cup\delta_T(t_4))\cup(\delta_S(s_5)\cup\delta_T(t_5))\\ &= \delta_{S\vee T}(s_1,t_1)\cup\delta_{S\vee T}(s_3,t_3)\cup\delta_{S\vee T}(s_5,t_5)\end{aligned}$$

Thus, $\delta_S\vee\delta_T$ is an S-uni BQI-ideal of $S\times T$. □

Note here that $\delta_S\wedge\delta_T$ is not always an S-uni BQI-ideal with Example 3.43.

Example 3.43. *Let's consider the semigroup in Example 3.2 and Example 3.12. Let* $\mathfrak{t}_S$ *and* $\mathfrak{t}_{\mathbb{R}}$ *be SS over* $U=Z_6=\{0,\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}$ *as follows:*

$$\mathfrak{t}_S=\{(z,\{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}),(\mathfrak{u},\{\bar{1},\bar{2},4\}),(\mathfrak{w},\{\bar{1},\bar{2},3\})\},\quad \mathfrak{t}_{\mathbb{R}}=\{(\mathfrak{c},\{\bar{1},\bar{4},\bar{5}\}),(\mathfrak{k},\{\bar{1},\bar{4},\bar{5}\})\}$$

$\mathfrak{t}_S$ *and* $\mathfrak{t}_{\mathbb{R}}$ *are S-uni BQI-ideals. Here, since*

$$\begin{aligned}\mathfrak{t}_{S\wedge\mathbb{R}}(z,\emptyset) &= \mathfrak{t}_{S\wedge\mathbb{R}}(\mathfrak{u},\emptyset)\cup\mathfrak{t}_{S\wedge\mathbb{R}}(\mathfrak{u},\emptyset)\cup\mathfrak{t}_{S\wedge\mathbb{R}}(z,\mathfrak{k})\\ &= \{\bar{3}\}\not\subseteq \mathfrak{t}_{S\wedge\mathbb{R}}(\mathfrak{u},\emptyset)\cup\mathfrak{t}_{S\wedge\mathbb{R}}(\mathfrak{u},\emptyset)\cup\mathfrak{t}_{S\wedge\mathbb{R}}(z,\mathfrak{k})=\{\bar{1},\bar{4},\bar{5}\}\end{aligned}$$

$\mathfrak{t}_S\wedge\mathfrak{t}_{\mathbb{R}}$ *is not an S-uni BQI-ideal.*

Theorem 3.44. *Let* ω_S *and* b_S *be S-uni BQI-ideals. Then,* $\omega_S\vee b_S$ *is an S-uni BQI-ideal.*

Proof. Let ω_S and b_S be S-uni BQI-ideals of S. Then,

$$\omega_S\star\widetilde{O}\star\omega_S\star\widetilde{O}\star\omega_S\ \tilde{\tilde{\supseteq}}\ \omega_S\quad\text{and}\quad b_S\star\widetilde{O}\star b_S\star\widetilde{O}\star b_S\ \tilde{\tilde{\supseteq}}\ b_S.$$

Thus,

$$(\omega_S\cup b_S)\star\widetilde{O}\star(\omega_S\cup b_S)\star\widetilde{O}\star(\omega_S\cup b_S)\ \tilde{\tilde{\supseteq}}\ \omega_S\star\widetilde{O}\star\omega_S\cup b_S\star\widetilde{O}\star b_S\ \tilde{\tilde{\supseteq}}\ \omega_S\cup b_S.$$

Hence, $(\omega_S\cup b_S)\star\widetilde{O}\star(\omega_S\cup b_S)\ \tilde{\tilde{\supseteq}}\ \omega_S\cup b_S$. Thus, $\omega_S\cup b_S$ is an S-uni BQI-ideal. □

Corollary 3.45. *Let* ω_S *be an S-uni* $\mathcal{R}$ *ideal (i.e., ideal, bi-ideal, interior-ideal, quasi-ideal, bi-interior, left bi-quasi, right bi-quasi, bi-quasi, left quasi-interior, right quasi-interior, or quasi-interior), and* b_S *be an S-uni* $\mathcal{R}$ *ideal. Then,* $\omega_S\cup b_S$ *is an S-uni BQI-ideal.*

Proposition 3.46. *Let* ω_S *be an S-uni* $\mathcal{L}$ *ideal and* b_S *be an SS. Then,* $\omega_S\star b_S$ *is an S-uni BQI-ideal.*

Proof. Let ω_S be S-uni $\mathcal{L}$ ideal of S. Then, $\widetilde{O}\star\omega_S\ \tilde{\tilde{\supseteq}}\ \omega_S$ and $\mathfrak{t}_S\star\mathfrak{t}_S\ \tilde{\tilde{\supseteq}}\ \mathfrak{t}_S$. Thus, $(\omega_S\star b_S)\star\widetilde{O}\star(\omega_S\star b_S)\star\widetilde{O}\star(\omega_S\star b_S)=\omega_S\star b_S\star(\widetilde{O}\star\omega_S)\star b_S\star(\widetilde{O}\star\omega_S)\star b_S\ \tilde{\tilde{\supseteq}}\ \omega_S\star b_S\star\omega_S\star b_S\ \tilde{\tilde{\supseteq}}\ \omega_S\star(\widetilde{O}\star\omega_S)\star(\widetilde{O}\star\omega_S)\star b_S\ \tilde{\tilde{\subseteq}}\ (\omega_S\star\omega_S)\star\omega_S\star b_S\ \tilde{\tilde{\supseteq}}\ \omega_S\star b_S$. Thus, $\mathfrak{t}_S\star b_S$ is an S-uni BQI-ideal. □

Proposition 3.47. *Let* ω_S *be an S-uni* $\mathcal{R}$ *ideal and* b_S *be an SS. Then,* $\omega_S\star b_S$ *is an S-uni BQI-ideal.*

Proof. Let ω_S be an S-uni $\mathcal{R}$ ideal of S. Then, $\omega_S\star\widetilde{O}\ \tilde{\tilde{\supseteq}}\ \omega_S$ and $\omega_S\star\omega_S\ \tilde{\tilde{\supseteq}}\ t_S$. Thus, $(\omega_S\star b_S)\star\widetilde{O}\star(\omega_S\star b_S)\ \tilde{\tilde{\supseteq}}\ (\omega_S\star\widetilde{O})\star(\omega_S\star b_S)\ \tilde{\tilde{\supseteq}}\ \omega_S\star b_S$. Thus, $\omega_S\star b_S$ is an S-uni BQI-ideal. □

Theorem 3.48. *Let* σ_S *be an S-uni ideal and* b_S *be an SS. Then,* $\sigma_S\star b_S$ *is an S-uni BQI-ideal.*

Proposition 3.49. *Let* b_S *be an S-uni* $\mathcal{L}$ *ideal and* σ_S *be an SS. Then,* $\sigma_S\star b_S$ *is an S-uni BQI-ideal.*

Proof. Let b_S be an S-uni $\mathcal{L}$ ideal of S. Then, $\widetilde{O} \star b_S \,\tilde{\tilde{\supseteq}}\, b_S$ and $b_S \star b_S \,\tilde{\tilde{\supseteq}}\, b_S$. Then,

$$\begin{aligned}(\sigma_S \star b_S) \star \widetilde{O} \star (\sigma_S \star b_S) \star \widetilde{O} \star (\sigma_S \star b_S) &\,\tilde{\tilde{\supseteq}}\, \sigma_S \star b_S \star (\widetilde{O} \star \sigma_S) \star b_S \star \sigma_S \star b_S \\ &\,\tilde{\tilde{\supseteq}}\, \sigma_S \star b_S.\end{aligned}$$

Thus, $\sigma_S \star b_S$ is an S-uni BQI-ideal. □

Proposition 3.50. *Let b_S be an S-uni $\mathcal{R}$ ideal and σ_S be an SS. Then, $\sigma_S \star b_S$ is an S-uni BQI-ideal.*

Proof. Let b_S be an S-uni $\mathcal{R}$ ideal of S. Then, $b_S \star \widetilde{O} \,\tilde{\tilde{\supseteq}}\, b_S$ and $b_S \star b_S \,\tilde{\tilde{\supseteq}}\, b_S$. Then,

$$(\sigma_S \star b_S) \star \widetilde{O} \star (\sigma_S \star b_S) \star \widetilde{O} \star (\sigma_S \star b_S) \,\tilde{\tilde{\supseteq}}\, \sigma_S \star (b_S \star \widetilde{O}) \star \sigma_S \star (b_S \star \widetilde{O}) \star \sigma_S \,\tilde{\tilde{\supseteq}}\, \sigma_S \star b_S.$$

Thus, $\sigma_S \star b_S$ is an S-uni BQI-ideal. □

Theorem 3.51. *Let b_S be an S-uni ideal and σ_S be an SS. Then, $\sigma_S \star b_S$ is an S-uni BQI-ideal.*

Proposition 3.52. *Let σ_S and b_S be SSs over U. If $\sigma_S \star b_S$ is an S-uni $\mathcal{L}$ ideal, then it is an S-uni BQI-ideal.*

Proof. Let $\mathsf{t}_S \star \mathsf{t}_S$ be an S-uni $\mathcal{L}$ ideal. Then,

$$(\sigma_S \star b_S) \star [\widetilde{O} \star (\sigma_S \star b_S)] \star [\widetilde{O} \star (\sigma_S \star b_S)] \,\tilde{\tilde{\supseteq}}\, [(\sigma_S \star b_S) \star (\sigma_S \star b_S)] \,\tilde{\tilde{\supseteq}}\, (\sigma_S \star b_S)$$

implying that $\mathsf{t}_S \star b_S$ is an S-uni BQI-ideal. □

Theorem 3.53. *Let t_S be an S-uni subsemigroup over U, α be a subset of U, $Im(\mathsf{t}_S)$ be the image of t_S such that $\alpha \in Im(\mathsf{t}_S)$. If t_S is an S-uni BQI-ideal of S, then $\mathcal{U}(\mathsf{t}_S;\alpha)$ is a BQI ideal of S.*

Proof. Since, $\mathsf{t}_S(x) = \alpha$ for some $x \in S$, $\emptyset \neq \mathcal{U}(\mathsf{t}_S;\alpha) \subseteq S$. Let $k \in \mathcal{U}(\mathsf{t}_S;\alpha) \cdot S \cdot \mathcal{U}(\mathsf{t}_S;\alpha)$, then there exist $x, y, z \in \mathcal{U}(\mathsf{t}_S;\alpha)$ and $b \in S$ such that $k = xbybz$. Thus, $\mathsf{t}_S(x) \subseteq \alpha$, $\mathsf{t}_S(y) \subseteq \alpha$ and $\mathsf{t}_S(z) \subseteq \alpha$. Since t_S is an S-uni BQI-ideal, $\mathsf{t}_S(k) = \mathsf{t}_S(xbybz) \subseteq \mathsf{t}_S(x) \cup \mathsf{t}_S(y) \cup \mathsf{t}_S(z) \subseteq \alpha \cup \alpha \cup \alpha = \alpha$. Hence, $\mathsf{t}_S(k) \subseteq \alpha$, implying that $k \in \mathcal{U}(\mathsf{t}_S;\alpha)$. Therefore, $\mathcal{U}(\mathsf{t}_S;\alpha) \cdot S \cdot \mathcal{U}(\mathsf{t}_S;\alpha) \subseteq \mathcal{U}(\mathsf{t}_S;\alpha)$. Moreover, since t_S is an S-uni subsemigroup over U, by Theorem 2.20, $\mathcal{U}(\mathsf{t}_S;\alpha)$ is a subsemigroup of S. Thus, $\mathcal{U}(\mathsf{t}_S;\alpha)$ is a BQI ideal. □

We illustrate Theorem 3.53 with Example 3.43.

Example 3.54. *Consider Example 3.2. It was shown in Example 3.2 that*

$$\mathsf{t}_S = \{(z, \{1, \overline{3}, 5, \overline{7}\}), (\ulcorner, \{1, \overline{3}, \overline{7}\}), (\ulcorner\ulcorner, \{1, \overline{3}\})\}$$

is an S-uni BQI-ideal. By considering the image set of t_S, that is,

$$Im(\mathsf{t}_S) = \{\{1, \overline{3}\}, \{1, \overline{3}, \overline{7}\}, \{1, \overline{3}, 5, \overline{7}\}\},$$

we obtain the following:

$$\mathcal{U}(\mathsf{t}_S;\alpha) = \begin{cases} \{\ulcorner\ulcorner\}, & \alpha = \{1, \overline{3}\} \\ \{\ulcorner, \ulcorner\ulcorner\}, & \alpha = \{1, \overline{3}, \overline{7}\} \\ \{z, \ulcorner, \ulcorner\ulcorner\}, & \alpha = \{1, \overline{3}, 5, \overline{7}\} \end{cases}$$

Here, $\{z, \ulcorner, \ulcorner\ulcorner\}$, $\{\ulcorner, \ulcorner\ulcorner\}$ and $\{\ulcorner\ulcorner\}$ are all BQI ideals of S. In fact, since

$\{z, \ulcorner, \ulcorner\ulcorner\} \cdot \{z, \ulcorner, \ulcorner\ulcorner\} \cdot \{z, \ulcorner, \ulcorner\ulcorner\} \subseteq \{z, \ulcorner, \ulcorner\ulcorner\}$, $\{\ulcorner, \ulcorner\ulcorner\} \cdot \{\ulcorner, \ulcorner\ulcorner\} \cdot \{\ulcorner, \ulcorner\ulcorner\} \subseteq \{\ulcorner, \ulcorner\ulcorner\}$, $\{\ulcorner\ulcorner\} \cdot \{\ulcorner\ulcorner\} \cdot \{\ulcorner\ulcorner\} \subseteq \{\ulcorner\ulcorner\}$

each $\mathcal{U}(\mathsf{t}_S;\alpha)$ is a subsemigroup of S. Similarly, since

$\{z, \ulcorner, \ulcorner\ulcorner\} \cdot S \cdot \{z, \ulcorner, \ulcorner\ulcorner\} \cdot S \cdot \{z, \ulcorner, \ulcorner\ulcorner\} \subseteq \{z, \ulcorner, \ulcorner\ulcorner\}$, $\{\ulcorner, \ulcorner\ulcorner\} \cdot S \cdot \{\ulcorner, \ulcorner\ulcorner\} \cdot S \cdot \{\ulcorner, \ulcorner\ulcorner\} \subseteq \{\ulcorner, \ulcorner\ulcorner\}$, $\{\ulcorner\ulcorner\} \cdot S \cdot \{\ulcorner\ulcorner\} \cdot S \cdot \{\ulcorner\ulcorner\} \subseteq \{\ulcorner\ulcorner\}$

each $\mathcal{U}(\mathsf{t}_S;\alpha)$ is a BQI ideal of S.

Now, consider the SS $6_S = \{(z, \{1, \overline{3}, 5, \overline{7}\}), (\ulcorner, \{1, \overline{3}, \overline{7}\}), (\ulcorner\ulcorner, \{1, \overline{3}, \overline{5}\})\}$ *in Example 3.2. By taking into account* $Im(6_S) = \{\{1, \overline{3}, \overline{5}\}, \{1, \overline{3}, \overline{7}\}, \{1, \overline{3}, 5, \overline{7}\}\}$ *we obtain the following:*

$$\mathcal{U}(6_S; \alpha) = \begin{cases} \{\ulcorner\ulcorner\}, & \alpha = \{1, \overline{3}, \overline{5}\} \\ \{\ulcorner\}, & \alpha = \{1, \overline{3}, \overline{7}\} \\ \{z, \ulcorner, \ulcorner\ulcorner\}, & \alpha = \{1, \overline{3}, 5, \overline{7}\} \end{cases}$$

Here, $\{\ulcorner\}$ *is not a BQI ideal of* S*. In fact, since*

$$\{\ulcorner\} \cdot S \cdot \{\ulcorner\} \cdot S \cdot \{\ulcorner\} = \{\ulcorner, \ulcorner\ulcorner\} \not\subseteq \{\ulcorner\}$$

one of the $\mathcal{U}(6_S; \alpha)$ *is not a BQI ideal of S, hence it is not a BQI ideal of* S*. It is seen that each of* $\mathcal{U}(6_S; \alpha)$ *is not a BQI ideal of S. On the other hand, in Example 3.2 it was shown that* 6_S *is not an S-uni BQI-ideal of S.*

Definition 3.55. *Let* $\mathfrak{t}_S$ *be an S-uni subsemigroup and S-uni BQI-ideal of* S*. Then, the BQI ideals* $\mathcal{U}(\mathfrak{t}_S; \alpha)$ *are called lower* α*-BQI ideals of* $\mathfrak{t}_S$*.*

Proposition 3.56. *Let* $\mathfrak{t}_S$ *be an SS over* U*,* $\mathcal{U}(\mathfrak{t}_S; \alpha)$ *be lower* α*-BQI of* $\mathfrak{t}_S$ *for each* $\alpha \subseteq U$ *and* $Im(\mathfrak{t}_S)$ *be an ordered set by inclusion. Then,* $\mathfrak{t}_S$ *is an S-uni BQI-ideal.*

Proof. Let $x, y, z, b, c \in S$ and $\mathfrak{t}_S(x) = \alpha_1$, $\mathfrak{t}_S(y) = \alpha_2$ and $\mathfrak{t}_S(z) = \alpha_3$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in \mathcal{U}(\mathfrak{t}_S; \alpha_1)$ and $y \in \mathcal{U}(\mathfrak{t}_S; \alpha_2)$. Since $\alpha_1 \subseteq \alpha_2 \subseteq \alpha_3$, $x, y, z \in \mathcal{U}(\mathfrak{t}_S; \alpha_1)$ and since $\mathcal{U}(\mathfrak{t}_S; \alpha)$ is a BQI of S for all $\alpha \subseteq U$, it follows that $xbycz \in \mathcal{U}(\mathfrak{t}_S; \alpha_1)$. Hence, $\mathfrak{t}_S(xbycz) \subseteq \alpha_1 \cup \alpha_2 \cup \alpha_3$. Thus, $\mathfrak{t}_S$ is an S-uni BQI-ideal. □

Proposition 3.57. *Let* $\mathfrak{t}_S$ *and* $\mathfrak{t}_T$ *be SS over* U*, and* Ψ *be a semigroup isomorphism from* S *to* T*. If* $\mathfrak{t}_S$ *is an S-uni BQI-ideal of* S*, then* $\Psi(\mathfrak{t}_S)$ *is an S-uni BQI-ideal of* T*.*

Proof. Let $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5 \in T$. Since Ψ is surjective, there exist $\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5 \in S$ such that $\Psi(\mathfrak{v}_1) = \mathfrak{t}_1$, $\Psi(\mathfrak{v}_2) = \mathfrak{t}_2$, $\Psi(\mathfrak{v}_3) = \mathfrak{t}_3$, $\Psi(\mathfrak{v}_4) = \mathfrak{t}_4$, and $\Psi(\mathfrak{v}_5) = \mathfrak{t}_5$. Then, $(\Psi(\mathfrak{t}_S))(\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4\mathfrak{t}_5) = \bigcap_{\mathfrak{v} \in S,\ \Psi(\mathfrak{v}) = \mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4\mathfrak{t}_5} \mathfrak{t}_S(\mathfrak{v}) = \bigcap_{\mathfrak{v} \in S,\ \mathfrak{v} = \Psi^{-1}(\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4\mathfrak{t}_5)} \mathfrak{t}_S(\mathfrak{v}) = \mathfrak{t}_S(\Psi^{-1}(\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4\mathfrak{t}_5)) = \mathfrak{t}_S(\mathfrak{v}_1\mathfrak{v}_2\mathfrak{v}_3\mathfrak{v}_4\mathfrak{v}_5)$, with $\mathfrak{v}_i \in S$, $\Psi(\mathfrak{v}_i) = \mathfrak{t}_i$, $i = 1, 2, 3, 4, 5$, so $\mathfrak{t}_S(\mathfrak{v}_1\mathfrak{v}_2\mathfrak{v}_3\mathfrak{v}_4\mathfrak{v}_5) \subseteq \mathfrak{t}_S(\mathfrak{v}_1) \cup \mathfrak{t}_S(\mathfrak{v}_3) \cup \mathfrak{t}_S(\mathfrak{v}_5)$, where $\Psi(\mathfrak{v}_1) = \mathfrak{t}_1$, $\Psi(\mathfrak{v}_3) = \mathfrak{t}_3$, $\Psi(\mathfrak{v}_5) = \mathfrak{t}_5$. Thus, $(\Psi(\mathfrak{t}_S))(\mathfrak{t}_1) \cup (\Psi(\mathfrak{t}_S))(\mathfrak{t}_3) \cup (\Psi(\mathfrak{t}_S))(\mathfrak{t}_5)$ Hence, $\Psi(\mathfrak{t}_S)$ is an S-uni BQI-ideal of T. □

Proposition 3.58. *Let* $\mathfrak{t}_S$ *and* $\mathfrak{t}_T$ *be SS over* U*, and* Ψ *be a semigroup isomorphism from* S *to* T*. If* $\mathfrak{t}_T$ *is an S-uni BQI-ideal of* S*, then* $\Psi^{-1}(\mathfrak{t}_T)$ *is an S-uni BQI-ideal of* T*.*

Proof. Let $\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5 \in S$. Then,

$$\begin{aligned} (\Psi^{-1}(\mathfrak{t}_T))(\mathfrak{v}_1\mathfrak{v}_2\mathfrak{v}_3\mathfrak{v}_4\mathfrak{v}_5) &= \mathfrak{t}_T(\Psi(\mathfrak{v}_1\mathfrak{v}_2\mathfrak{v}_3\mathfrak{v}_4\mathfrak{v}_5)) \\ &= \mathfrak{t}_T(\Psi(\mathfrak{v}_1)\Psi(\mathfrak{v}_2)\Psi(\mathfrak{v}_3)\Psi(\mathfrak{v}_4)\Psi(\mathfrak{v}_5)) \\ &\subseteq \mathfrak{t}_T(\Psi(\mathfrak{v}_1)) \cup \mathfrak{t}_T(\Psi(\mathfrak{v}_3)) \cup \mathfrak{t}_T(\Psi(\mathfrak{v}_5)) \\ &= (\Psi^{-1}(\mathfrak{t}_T))(\mathfrak{v}_1) \cup (\Psi^{-1}(\mathfrak{t}_T))(\mathfrak{v}_3) \cup (\Psi^{-1}(\mathfrak{t}_T))(\mathfrak{v}_5) \end{aligned}$$

Thus, $\Psi^{-1}(\mathfrak{t}_T)$ is an S-uni BQI-ideal of T. □

Theorem 3.59. *For a semigroup* S*, the following conditions are equivalent:*

(1) S *is an* $\mathcal{R}$ *semigroup.*
(2) $\mathfrak{t}_S = \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S$ *for every S-uni BQI-ideal of* S*.*

Proof. First assume that (1) holds. Let S be an $\mathcal{R}$ semigroup, $\mathfrak{t}_S$ be an S-uni BQI-ideal of S and $\mathfrak{m} \in S$. Then, $\mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \mathrel{\tilde{\subseteq}} \mathfrak{t}_S$ and there exists an element $\mathfrak{n} \in S$ such that $\mathfrak{m} = \mathfrak{n}\mathfrak{n}\mathfrak{n}$.

Thus,

$$\begin{aligned}
(\mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S)(\mathfrak{m}) &= \bigcap_{\mathfrak{m}=\mathfrak{l}\mathfrak{n}} \left\{(\mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \star \tilde{\mathcal{O}})(\mathfrak{l}) \cup \mathfrak{t}_S(\mathfrak{n})\right\} \\
&\subseteq (\mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \star \tilde{\mathcal{O}})(\mathfrak{m}\mathfrak{n}) \cup \mathfrak{t}_S(\mathfrak{m}) \\
&= \bigcap_{\mathfrak{m}\mathfrak{n}=pq} \left\{(\mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S)(p) \cup \tilde{\mathcal{O}}(q)\right\} \cup \mathfrak{t}_S(\mathfrak{m}) \\
&\subseteq (\mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S)(\mathfrak{m}) \cup \tilde{\mathcal{O}}(\mathfrak{n}\mathfrak{n}\mathfrak{n}) \cup \mathfrak{t}_S(\mathfrak{m}) \\
&= \bigcap_{\mathfrak{m}=\mathfrak{l}\mathfrak{n}} \left\{(\mathfrak{t}_S \star \tilde{\mathcal{O}})(\mathfrak{l}) \cup \mathfrak{t}_S(\mathfrak{n})\right\} \cup \mathfrak{t}_S(\mathfrak{m}) \\
&\subseteq (\mathfrak{t}_S \star \tilde{\mathcal{O}})(\mathfrak{m}\mathfrak{n}) \cup \mathfrak{t}_S(\mathfrak{m}) \cup \mathfrak{t}_S(\mathfrak{m}) \\
&= \bigcap_{\mathfrak{m}\mathfrak{n}=pq} \left\{\mathfrak{t}_S(p) \cup \tilde{\mathcal{O}}(q)\right\} \cup \mathfrak{t}_S(\mathfrak{m}) \\
&= \mathfrak{t}_S(\mathfrak{m}) \cup \tilde{\mathcal{O}}(\mathfrak{n}\mathfrak{n}\mathfrak{n}) \cup \mathfrak{t}_S(\mathfrak{m}) \\
&= \mathfrak{t}_S(\mathfrak{m})
\end{aligned}$$

Therefore, $\mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \,\tilde{\subseteq}\, \mathfrak{t}_S$ implying that $\mathfrak{t}_S = \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S$.
Conversely, let $\mathfrak{t}_S = \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S \star \tilde{\mathcal{O}} \star \mathfrak{t}_S$ where $\mathfrak{t}_S$ is an S-uni BQI-ideal of S. In order to show that S is an $\mathcal{R}$ semigroup, we need to show that $A = ASASA$ for every BQI ideal A of S. It is obvious that $ASASA \subseteq A$. Thus, it is enough to show that $A \subseteq ASASA$. Let there exist $\mathfrak{l}_0 \in A$ such that $\mathfrak{l}_0 \notin ASASA$. By Theorem 2.10, S_{A^c} is an S-uni BQI of S. Since $\mathfrak{l}_0 \in A$, $S_{A^c}(\mathfrak{l}_0) = \varnothing$. On the other hand, since $\mathfrak{l}_0 \notin ASASA$, this implies that there does not exist $\mathfrak{n}, \mathfrak{m}, p \in A$ and $q, s \in S$ such that $\mathfrak{l}_0 = \mathfrak{n}q\mathfrak{m}sp$. Thus, $(S_{A^c} \star \tilde{\mathcal{O}} \star S_{A^c} \star \tilde{\mathcal{O}} \star S_{A^c})(\mathfrak{l}_0) = (S_{A^c} \star S_{S^c} \star S_{A^c} \star S_{S^c} \star S_{A^c})(\mathfrak{l}_0) = S_{A^c}(\mathfrak{l}_0) = U$. However, this conflicts with our assertion. Thus, $\mathfrak{l}_0 \in ASASA$, $A = ASASA$, and so S is an $\mathcal{R}$ semigroup. ☐

4. Conclusion

In this study, the concept of the soft union (S-uni) bi-quasi-interior (BQI) ideal in semigroups is proposed and the relations of several types of S-uni ideals with S-uni BQI ideals are provided. It is obtained that every S-uni bi-ideal, S-uni ideal, S-uni interior ideal, S-uni quasi-ideal, S-uni bi-interior ideal, S-uni bi-quasi ideal, and S-uni quasi-interior ideal of a semigroup is an S-uni BQI ideal, however, the converses are not true. The conditions for the converses to hold are also explored. Moreover, it is shown that if a subsemigroup of a semigroup is a BQI ideal, then its soft characteristic function is also an S-uni BQI ideal, and the converse is also true. This result highlights the significant connection between classical semigroup theory and soft set theory. Future work could explore more characterizations of an S-uni BQI ideal with certain types of semigroups like intra-regular, weakly-regular, quasi-regular, semisimple and duo semigroups.

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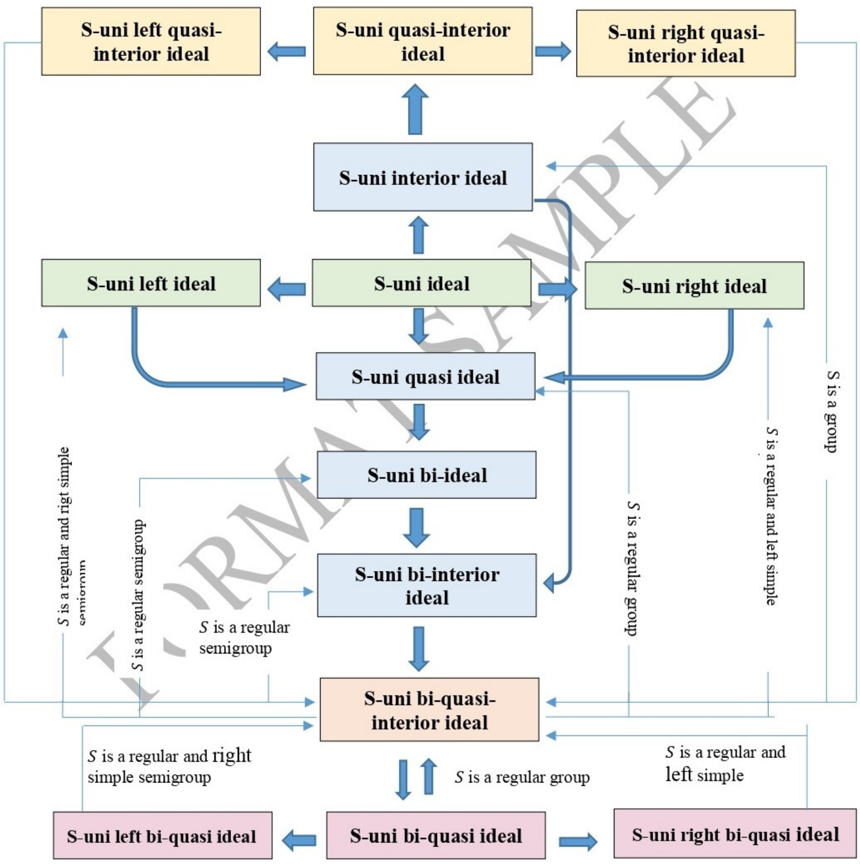
Appendix

The relation between several S-uni ideals and their generalized ideals is depicted in the following figure, where ₭ → ₱ denotes that ₭ is ₱ but ₱ may not always be ₭.



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